

# MINKOWSKI CONTENT AND FRACTAL EULER CHARACTERISTIC FOR CONFORMAL GRAPH DIRECTED SYSTEMS

MARC KESSEBÖHMER AND SABRINA KOMBRINK

**ABSTRACT.** We show that the fractal Euler characteristic and the Minkowski content of a limit set of a conformal graph directed system, which consists of similarities, exist if and only if the associated geometric potential function is non-lattice. This extends a result by M. L. Lapidus and M. van Frankenhuysen for non-degenerate self-similar subsets of  $\mathbb{R}$  that satisfy the open set condition with connected feasible open set. We moreover generalise to systems consisting of conformal maps and in particular obtain that limit sets of Fuchsian groups of Schottky type are always Minkowski measurable. This proves a conjecture of M. L. Lapidus from 1993. We additionally gain results on the existence of local versions of the fractal Euler characteristic and the Minkowski content of limit sets of conformal graph directed systems. These local versions turn out to be constant multiples of the  $\delta$ -conformal measure, whenever they exist, where  $\delta$  denotes the Minkowski dimension of the limit set.

## 1. INTRODUCTION

In [KK12] the authors examined the existence of the Minkowski content and of the fractal Euler characteristic for self-conformal subsets of  $\mathbb{R}$ , that are sets which arise as the invariant sets of conformal iterated function systems (cIFS), see Definition 2.8. These studies are continued in this present article. We consider the Minkowski content and the fractal Euler characteristic for limit sets of finite conformal graph directed systems (cGDS), that are embedded in  $\mathbb{R}$ , as introduced for instance in [MU03]. The class of cGDS generalises the class of cIFS that was studied in [KK12] and gives rise to a much richer collection of fractal sets. Sets which belong to the former class but not to the latter include limit sets of Fuchsian groups of Schottky type, limit sets of Markov interval maps and invariant sets of cIFS satisfying the open set condition (OSC) with disconnected feasible open sets (see Section 4 for more details on these and other examples).

Previous to [KK12] mainly self-similar fractals were investigated with respect to the existence of the Minkowski content. One important result for such sets is that a non-degenerate self-similar subset of  $\mathbb{R}$  that satisfies the open set condition with connected feasible open set is Minkowski measurable if and only if the associated geometric potential function is non-lattice [LP93, Fal95, LvF06]. We significantly extend this result and provide an alternative proof by showing that the analogous statement is true also in the graph directed setting. To be more precise,

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we obtain that the limit set of a cGDS that consists of similarities (sGDS) and is non-degenerate is Minkowski measurable if and only if the sGDS is non-lattice (Corollary 3.17). This convenient equivalence statement for systems consisting of similarities unfortunately fails to hold for general conformal systems: In [KK12] it was shown that there exist non-degenerate self-conformal sets arising from lattice cIFS for which the Minkowski content and the fractal Euler characteristic exist. Since self-conformal sets are special types of limit sets of cGDS we cannot expect the equivalence to be valid for general limit sets of cGDS either. In Theorem 3.14 we provide a sufficient condition for the Minkowski content and the fractal Euler characteristic of a limit set of a cGDS to exist in the lattice case and moreover prove existence in the non-lattice situation. Furthermore, we show in Theorem 3.14 that an average version of the Minkowski content always exists and provide an explicit formula to determine its value.

The geometric potential function of a Fuchsian group of Schottky type is non-lattice, and thus we obtain as a corollary to Theorem 3.14 that the Minkowski content of a limit set of a Fuchsian group of Schottky type always exists (see Section 4.5). This result proves a conjecture by M. L. Lapidus from 1993 [Lap93], which plays an important role in the context of the Weyl-Berry conjecture. The Weyl-Berry conjecture for fractal drums is a conjecture on the distribution of the eigenvalues of the Laplacian on a domain with a fractal boundary (see [LP93, Lap93, Fal95]). It addresses the problem of describing ‘the relationship between the shape (geometry) of the drum and its sound (its spectrum).’ [LvF06, p.1] A more detailed exposition on the results from the literature and on the above mentioned conjecture will be given in Remark 3.22.

The Weyl-Berry conjecture is one of the main motivations for studying the Minkowski content, see for instance [Lap93, LvF06, Kom12]. A second motivation for studying the Minkowski content of a fractal set arises from non-commutative geometry: In Connes’ seminal book [Con94] the notion of a non-commutative fractal geometry is developed. There it is shown that the natural analogue of the volume of a compact smooth Riemannian spin<sup>c</sup> manifold for a fractal set in  $\mathbb{R}$  is that of the Minkowski content. This idea is also reflected in [GI03, Sam10, FS11].

Another main motivation for studying the Minkowski content arises from fractal geometry, where one aims to find characteristics that describe the geometric structure of a fractal set. The Minkowski content can be viewed as such a tool. It complements the notion of dimension and is capable of distinguishing between sets of the same Hausdorff- or Minkowski dimension. More precisely, considering two fractal sets  $F_1, F_2 \subseteq [0, 1]$  with  $\{0, 1\} \subseteq F_1, F_2$  which are of the same Minkowski dimension, the Minkowski content compares the rate of decay of the lengths of the  $\varepsilon$ -parallel neighbourhoods of  $F_1$  and  $F_2$ . In this way it can be interpreted as “fractal length”. Also the fractal Euler characteristic can be viewed as a characteristic describing the geometric structure of a fractal set beyond its dimension. It was first introduced and examined in [LW07] and was further investigated in the context of fractal curvature measures in [Win08]. If the ambient space is of dimension one, then there are two fractal curvature measures: The 0-th fractal curvature measure is a localised version of the fractal Euler characteristic (which we call the local fractal Euler characteristic) and the 1-st fractal curvature measure is a localised version of the Minkowski content (which we call the local Minkowski content). The term “curvature” is appropriate for higher dimensional ambient spaces but strictly

speaking not for one-dimensional spaces. However, we will sometimes use the term fractal curvature measures to refer to both the local Euler characteristic and the local Minkowski content. The local Minkowski content and the local fractal Euler characteristic are Borel-measures which describe the “fractal length” and “fractal Euler characteristic” of a given fractal inside of a Borel set. We obtain that these measures exist for limit sets of non-lattice cGDS and are constant multiples of the associated  $\delta$ -conformal measure, where  $\delta$  denotes the Minkowski dimension of the limit set (see Theorem 3.13). For limit sets of lattice sGDS we prove that these measures do not exist (see Theorem 3.16). They neither exist for  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of limit sets of lattice sGDS (see Theorem 3.18). This latter statement is important to note, since there exist  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of limit sets of lattice sGDS for which the Minkowski content and the fractal Euler characteristic do exist (see Example 4.4). Also for limit sets of lattice cGDS consisting of analytic maps, the local Minkowski content and the local fractal Euler characteristic do not exist (see Theorem 3.13). However, we show that in the lattice situation average versions of the local Minkowski content and the local fractal Euler characteristic of a limit set of a cGDS always exist and are constant multiples of the associated  $\delta$ -conformal measure, where  $\delta$  denotes the Minkowski dimension of the limit set (see Theorem 3.13). For an overview of the relevant literature and more background on the (local) Minkowski content and the (local) fractal Euler characteristic, we refer the reader to [KK12, Kom12].

We also remark that there are several recent articles dealing with the existence of the Minkowski content in higher dimensional ambient spaces. An important contribution is provided by [Gat00], where it is shown that the Minkowski content of self-similar sets arising from non-lattice IFS that satisfy the OSC exists. Alternative proofs of this result and further investigations on the lattice case are provided in [DKO<sup>+</sup>10, LPW11]. Minkowski measurability of self-conformal sets in higher dimensional ambient spaces has been studied in [Kom11]. There it is shown that, under certain geometric conditions, a self-conformal set arising from a non-lattice cIFS is Minkowski measurable. For an overview of the recent development of this research area we refer the reader to the survey article [Kom12].

This article is organised as follows. In Section 2 we give the construction of cGDS and their limit sets. In Section 3 we present our main results on the existence of the Minkowski content, the fractal Euler characteristic and their local versions. Section 4 is devoted to demonstrating how the new results can be applied to various classes of examples of limit sets of cGDS. Sections 5 to 7 deal with the proofs of the main theorems. More precisely, in Section 5 we provide some background and prove auxiliary results. With this knowledge we provide the proofs of our main theorems concerning limit sets of cGDS (Theorems 3.13 and 3.14) in Section 6. Section 7 deals with the proofs of Theorems 3.16 and 3.18, which are concerned with the special cases of sGDS as well as  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of limit sets of sGDS.

## 2. CONFORMAL GRAPH DIRECTED SYSTEMS

A core text concerning conformal graph directed systems (cGDS) is [MU03]. The class of cGDS generalises the notion of conformal iterated function systems and gives rise to a much richer class of fractal sets such as limit sets of Fuchsian groups. In Section 4 we give examples of classes of fractal sets which can be obtained via a cGDS. In this section, we present the relevant definitions.

**Definition 2.1** (Directed multigraph). A *directed multigraph*  $(V, E, i, t)$  consists of a finite set of vertices  $V$ , a finite set of directed edges  $E$  and functions  $i, t: E \rightarrow V$  which determine the initial and terminal vertex of an edge. The edge  $e \in E$  goes from  $i(e)$  to  $t(e)$ . Thus, the *initial* and *terminal vertices* of  $e$  are  $i(e)$  and  $t(e)$  respectively.

**Definition 2.2** (Incidence matrix). Given a directed multigraph  $(V, E, i, t)$ , an  $(\#E) \times (\#E)$ -matrix  $A$  with entries in  $\{0, 1\}$ , which satisfies  $A_{e,e'} = 1$  if and only if  $t(e) = i(e')$  for edges  $e, e' \in E$ , is called an *incidence matrix*. The incidence matrix  $A$  is called *aperiodic and irreducible* if there exists an  $n \in \mathbb{N}$  such that the entries of the  $n$ -folded product  $A^n$  are all positive.

**Definition 2.3** (GDS). A *graph directed system* (GDS) consists of a directed multigraph  $(V, E, i, t)$  with incidence matrix  $A$ , a family of non-empty compact connected metric spaces  $(X_v)_{v \in V}$  and for each edge  $e \in E$  an injective contraction  $\phi_e: X_{t(e)} \rightarrow X_{i(e)}$  with Lipschitz constant less than or equal to  $r$  for some  $r \in (0, 1)$ . Briefly, the family  $\Phi := (\phi_e: X_{t(e)} \rightarrow X_{i(e)})_{e \in E}$  is called a GDS.

In this paper, we consider fractal subsets of the real line. Therefore, we restrict the definition of a cGDS to the one-dimensional Euclidean space  $(\mathbb{R}, |\cdot|)$ . For a subset  $Y$  of  $(\mathbb{R}, |\cdot|)$  we let  $\text{int}(Y)$  denote its interior and  $\overline{Y}$  its closure.

**Definition 2.4** (cGDS). A GDS is called *conformal* (cGDS) if

- (i) for every vertex  $v \in V$ ,  $X_v$  is a compact connected subset of  $(\mathbb{R}, |\cdot|)$  satisfying  $X_v = \overline{\text{int}(X_v)}$ ,
- (ii) the *open set condition* (OSC) is satisfied, in the sense that, for all  $e \neq e' \in E$  we have

$$\phi_e(\text{int}(X_{t(e)})) \cap \phi_{e'}(\text{int}(X_{t(e')})) = \emptyset \quad \text{and}$$

- (iii) for every vertex  $v \in V$  there exists an open connected set  $W_v \supset X_v$  such that for every  $e \in E$  with  $t(e) = v$  the map  $\phi_e$  extends to a  $\mathcal{C}^{1+\alpha}$ -diffeomorphism from  $W_v$  into  $W_{i(e)}$ , whose derivative  $\phi'_e$  is bounded away from zero on  $W_v$ , where  $\alpha \in (0, 1]$ .

We also consider the special case of cGDS where the contractions  $\phi_e$  for  $e \in E$  are similarities:

**Definition 2.5** (sGDS). A cGDS, whose maps  $\phi_e$  are similarities for  $e \in E$ , is referred to as sGDS.

*Remark 2.6.* In the sequel, we will often refer to results from [MU03], where conformal graph directed Markov systems (cGDMS) are treated. Such systems differ from cGDS in the sense of Definition 2.4 in the following way. Firstly, an incidence matrix for a cGDMS only fulfils the property that  $A_{e,e'} = 1$  implies  $t(e) = i(e')$ . Secondly, we require the contractions  $\phi_e$  for  $e \in E$  to extend to  $\mathcal{C}^{1+\alpha}$ -diffeomorphisms with derivatives bounded away from zero, whereas for a cGDMS the contractions need to extend to  $\mathcal{C}^1$ -diffeomorphisms and are required to satisfy a bounded distortion property. However, every cGDS in our sense is a cGDMS in the sense of [MU03]. Conversely, disregarding the second difference, a cGDMS in  $\mathbb{R}$  can always be represented by a cGDS in our sense, namely by substituting  $(\phi_e(X_{t(e)}))_{e \in E}$  in for the sets  $(X_v)_{v \in V}$  and defining the edges accordingly.

In order to define the limit set of a cGDS, we fix a cGDS with the notation from Definitions 2.3 and 2.4. The set of *infinite admissible words* given by the incidence matrix  $A$  is defined to be

$$(2.1) \quad E_A^\infty := \{\omega = \omega_1\omega_2 \cdots \in E^\mathbb{N} \mid A_{\omega_n, \omega_{n+1}} = 1 \text{ for all } n \in \mathbb{N}\}.$$

The set of sub-words of length  $n \in \mathbb{N}$  is denoted by  $E_A^n$  and the set of all finite sub-words including the empty word  $\emptyset$  by  $E_A^*$ . For a finite word  $\omega \in E_A^*$  we let  $n(\omega)$  denote its length, where  $n(\emptyset) := 0$ , define  $\phi_\emptyset$  to be the identity map on  $\bigcup_{v \in V} X_v$  and for  $\omega \in E_A^* \setminus \{\emptyset\}$  set

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_{n(\omega)}} : X_{t(\omega_{n(\omega)})} \rightarrow X_{i(\omega_1)},$$

where we let  $\omega_i$  denote the  $i$ -th letter of the word  $\omega$  for  $i \in \{1, \dots, n(\omega)\}$ , that is  $\omega = \omega_1 \cdots \omega_{n(\omega)}$ . For two finite words  $u = u_1 \cdots u_n$ ,  $\omega = \omega_1 \cdots \omega_m \in E_A^*$ , we let  $u\omega := u_1 \cdots u_n \omega_1 \cdots \omega_m \in E_A^*$  denote their concatenation. Likewise, we set  $u\omega := u_1 \cdots u_n \omega_1 \omega_2 \cdots$  for  $u = u_1 \cdots u_n \in E_A^*$  and  $\omega = \omega_1 \omega_2 \cdots \in E_A^\infty$ . For an infinite word  $\omega = \omega_1 \omega_2 \cdots \in E_A^\infty$  and  $n \in \mathbb{N}$  the *initial word of length  $n$*  is defined to be  $\omega|_n := \omega_1 \cdots \omega_n$ .

For  $\omega \in E_A^\infty$  the sequence of sets  $(\phi_{\omega|_n}(X_{t(\omega_n)}))_{n \in \mathbb{N}}$  form a descending sequence of non-empty compact sets and therefore  $\bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)}) \neq \emptyset$ . Recall from Definition 2.3 that we let  $r \in (0, 1)$  denote a common Lipschitz constant of the functions  $\phi_e$  for  $e \in E$ . Since  $\text{diam}(\phi_{\omega|_n}(X_{t(\omega_n)})) \leq r^n \text{diam}(X_{t(\omega_n)}) \leq r^n \max\{\text{diam}(X_v) \mid v \in V\}$  for every  $n \in \mathbb{N}$ , the intersection

$$\bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton and we denote its only element by  $\pi(\omega)$ . The map  $\pi : E_A^\infty \rightarrow \bigcup_{v \in V} X_v$  is called the *code map*.

**Definition 2.7** (Limit set of a cGDS). The *limit set* of the cGDS  $(\phi_e)_{e \in E}$  is defined to be

$$F := \pi(E_A^\infty).$$

Limit sets of cGDS often have a fractal structure. They include invariant sets of conformal iterated function systems, the so-called self-conformal sets, as well as self-similar sets. These are defined as follows.

**Definition 2.8** (cIFS, self-conformal set, self-similar set). A *conformal iterated function system* (cIFS) is a cGDS  $\Psi := (\psi_1, \dots, \psi_N)$  whose set of vertices  $V$  is a singleton and whose set of edges contains at least two elements. The unique limit set of a cIFS is called the *self-conformal set* associated with  $\Psi$ . In the case that the maps  $\psi_1, \dots, \psi_N$  are similarities, the limit set is called the *self-similar set* associated with  $\Psi$ .

In order to show the significance of cGDS, Section 4 is devoted to examples of important classes of such sets.

### 3. MAIN RESULTS

**3.1. Notation, Definitions and First Results.** Before stating our results, let us begin with recalling the relevant notations and definitions, in particular the local Minkowski content and the local fractal Euler characteristic. For further background we refer the reader to [KK12]. We let  $\lambda^0$  and  $\lambda^1$  respectively denote

the counting measure and the one-dimensional Lebesgue measure. For  $\varepsilon > 0$  we define  $Y_\varepsilon := \{x \in \mathbb{R} \mid \inf_{y \in Y} |x - y| \leq \varepsilon\}$  to be the  $\varepsilon$ -parallel neighbourhood of  $Y \subset \mathbb{R}$  and let  $\partial Y$  denote the boundary of  $Y$ .

**Definition 3.1** (Scaling exponents). For a non-empty compact set  $Y \subset \mathbb{R}$  the  $0$ -th and  $1$ -st curvature scaling exponents of  $Y$  are respectively defined to be

$$s_0(Y) := \inf\{t \in \mathbb{R} \mid \varepsilon^t \lambda^0(\partial Y_\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0\} \quad \text{and} \\ s_1(Y) := \inf\{t \in \mathbb{R} \mid \varepsilon^t \lambda^1(Y_\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0\}.$$

**Definition 3.2** (Local fractal Euler characteristic, local Minkowski content). Let  $Y \subset \mathbb{R}$  denote a non-empty compact set. Provided, that the weak limit

$$C_0^f(Y, \cdot) := \text{w-lim}_{\varepsilon \rightarrow 0} \varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap \cdot) / 2$$

of the finite Borel measures  $\varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap \cdot) / 2$  exists, we call it the *local fractal Euler characteristic* of  $Y$ . Likewise, the weak limit

$$C_1^f(Y, \cdot) := \text{w-lim}_{\varepsilon \rightarrow 0} \varepsilon^{s_1(Y)} \lambda^1(Y_\varepsilon \cap \cdot)$$

is called the *local Minkowski content* of  $Y$ , if it exists. Moreover, for a Borel set  $B \subseteq \mathbb{R}$  we set

$$\overline{C}_0^f(Y, B) := \limsup_{\varepsilon \rightarrow 0} \varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap B) / 2, \quad \overline{C}_1^f(Y, B) := \limsup_{\varepsilon \rightarrow 0} \varepsilon^{s_1(Y)} \lambda^1(Y_\varepsilon \cap B), \\ \underline{C}_0^f(Y, B) := \liminf_{\varepsilon \rightarrow 0} \varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap B) / 2, \quad \underline{C}_1^f(Y, B) := \liminf_{\varepsilon \rightarrow 0} \varepsilon^{s_1(Y)} \lambda^1(Y_\varepsilon \cap B).$$

*Remark 3.3.* The fractal Euler characteristic was investigated first in [LW07]. In higher dimensional ambient spaces, the local fractal Euler characteristic and the local Minkowski content belong to the class of fractal curvature measures as introduced by S. Winter in [Win08]. Although the notion of curvature is appropriate only in higher dimensional ambient spaces, we will use the terminology from higher dimensions and refer to the fractal Euler characteristic and the local Minkowski content as the *0-th and 1-st fractal curvature measure* respectively.

We will see that the fractal curvature measures of limit sets of cGDS do not always exist. In these cases, however, the following average versions do exist.

**Definition 3.4** (Average local fractal Euler characteristic, average local Minkowski content). Let  $Y \subset \mathbb{R}$  denote a non-empty compact set. Provided that the weak limit exists, we call

$$\tilde{C}_0^f(Y, \cdot) := \text{w-lim}_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{s_0(Y)-1} \lambda^0(\partial Y_\varepsilon \cap \cdot) d\varepsilon / 2$$

the *average local fractal Euler characteristic* of  $Y$  (or the *0-th average fractal curvature measure* of  $Y$ ) and let the weak limit

$$\tilde{C}_1^f(Y, \cdot) := \text{w-lim}_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{s_1(Y)-1} \lambda^1(Y_\varepsilon \cap \cdot) d\varepsilon$$

denote the *average local Minkowski content* of  $Y$  (or the *1-st average fractal curvature measure* of  $Y$ ).

*Remark 3.5.* If  $C_k^f(Y, \cdot)$  exists, then also  $\tilde{C}_k^f(Y, \cdot)$  exists and the two signed Borel measures coincide.

The definition of the 1-st curvature scaling exponent resembles the definition of the Minkowski dimension, which coincides with the box counting dimension (see [Fal03, Proposition 3.2]), and is defined next.

**Definition 3.6** ((Upper and lower) Minkowski dimension). For a non-empty compact set  $Y \subset \mathbb{R}$  the *upper* and *lower Minkowski dimensions* are respectively defined by

$$\overline{\dim}_M(Y) := 1 - \liminf_{\varepsilon \searrow 0} \frac{\ln \lambda^1(Y_\varepsilon)}{\ln \varepsilon} \quad \text{and} \quad \underline{\dim}_M(Y) := 1 - \limsup_{\varepsilon \searrow 0} \frac{\ln \lambda^1(Y_\varepsilon)}{\ln \varepsilon}.$$

In the case that the upper and lower Minkowski dimensions coincide, we call the common value the *Minkowski dimension* of  $Y$  and denote it by  $\dim_M(Y)$ .

For limit sets of cGDS with aperiodic irreducible incidence matrix the Minkowski dimension always exists (see Theorem 5.6). Moreover, as we will see, such a limit set is either a non-empty compact interval or has one-dimensional Lebesgue measure 0 (see Proposition 5.1). In order to determine the fractal curvature scaling exponents we have to distinguish between these two cases.

**Proposition 3.7.** *Let  $\delta$  denote the Minkowski dimension of the limit set  $F$  of a cGDS. If  $\lambda^1(F) = 0$ , then  $s_0(F) = \delta$  and  $s_1(F) = \delta - 1$ . If  $F$  is a non-empty compact interval, then  $s_0(F) = s_1(F) = 0$ .*

Let us first consider the latter situation of the above proposition. In this case, as an immediate consequence of Proposition 3.7, we obtain the following complete description.

**Corollary 3.8.** *If  $Y \subset \mathbb{R}$  is a non-empty compact interval, then both the 0-th and 1-st fractal curvature measures exist and satisfy*

$$C_0^f(Y, \cdot) = \lambda^0(\partial Y \cap \cdot)/2 \quad \text{and} \quad C_1^f(Y, \cdot) = \lambda^1(Y \cap \cdot).$$

Let us now focus on limit sets with one-dimensional Lebesgue measure 0. Here, the total mass of the 1-st (average) fractal curvature measure is given by the (average) Minkowski content:

**Definition 3.9** ((Upper and lower) Minkowski content, average Minkowski content). Let  $Y \subset \mathbb{R}$  denote a set for which the Minkowski dimension  $\dim_M(Y)$  exists. The *upper Minkowski content*  $\overline{\mathcal{M}}(Y)$  and the *lower Minkowski content*  $\underline{\mathcal{M}}(Y)$  of  $Y$  are defined to be

$$\overline{\mathcal{M}}(Y) := \limsup_{\varepsilon \rightarrow 0} \varepsilon^{\dim_M(Y)-1} \lambda^1(Y_\varepsilon) \quad \text{and} \quad \underline{\mathcal{M}}(Y) := \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\dim_M(Y)-1} \lambda^1(Y_\varepsilon).$$

If the upper and lower Minkowski contents coincide, then we denote the common value by  $\mathcal{M}(Y)$  and call it the *Minkowski content* of  $Y$ . In the case that the Minkowski content exists, is positive and finite, we call  $Y$  *Minkowski measurable*. The *average Minkowski content* of  $Y$  is defined to be the following limit, provided it exists

$$\widetilde{\mathcal{M}}(Y) := \lim_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2} \lambda^1(Y_\varepsilon) d\varepsilon.$$

**Remark 3.10.** If  $C_1^f(Y, \cdot)$  exists, then  $\mathcal{M}(Y)$  exists and  $C_1^f(Y, \mathbb{R})$  and  $\mathcal{M}(Y)$  coincide. The same connection holds for the lower, upper and average versions.



For stating our results on limit sets of one-dimensional Lebesgue measure 0, we fix a cGDS  $(V, E, i, t, A)$  and assume that the incidence matrix  $A$  is aperiodic and irreducible (see Definition 2.2). Let  $(X_v)_{v \in V}$  denote the associated non-empty compact connected subsets of  $\mathbb{R}$  and let  $\Phi := (\phi_e: X_{t(e)} \rightarrow X_{i(e)})_{e \in E}$  denote the family of injective  $r$ -Lipschitz maps for some  $r \in (0, 1)$ . Further, let  $F$  denote the unique limit set and let  $\delta := \dim_M(F)$  be its Minkowski dimension. A central role with regard to our results is played by the geometric potential function:

**Definition 3.11** (Geometric potential function, shift-map). The *geometric potential function*  $\xi: E_A^\infty \rightarrow \mathbb{R}$  is defined by  $\xi(\omega) := -\ln|\phi'_{\omega_1}(\pi(\sigma\omega))|$  for  $\omega = \omega_1\omega_2\cdots \in E_A^\infty$ . Here  $\sigma: E_A^* \cup E_A^\infty \rightarrow E_A^* \cup E_A^\infty$  denotes the *shift-map* which is defined by  $\sigma(\omega) := \emptyset$  for  $\omega \in \{\emptyset\} \cup E_A^1$ ,  $\sigma(\omega_1\cdots\omega_n) := \omega_2\cdots\omega_n \in E_A^{n-1}$  for  $\omega_1\cdots\omega_n \in E_A^n$ , where  $n \geq 2$  and  $\sigma(\omega_1\omega_2\cdots) := \omega_2\omega_3\cdots \in E_A^\infty$  for  $\omega_1\omega_2\cdots \in E_A^\infty$ .

We equip  $E^\mathbb{N}$  with the product topology of the discrete topologies on  $E$  and equip the set of infinite admissible words  $E_A^\infty \subset E^\mathbb{N}$  with the subspace topology. This is the weakest topology with respect to which the canonical projections onto the coordinates are continuous. The space of continuous real-valued functions on  $E_A^\infty$  is denoted by  $\mathcal{C}(E_A^\infty)$ . Note that the geometric potential function  $\xi$  belongs to  $\mathcal{C}(E_A^\infty)$ . A crucial property of the geometric potential function is whether it is lattice or non-lattice.

**Definition 3.12** (Cohomologous, lattice, non-lattice). (i) Two functions  $f_1, f_2 \in \mathcal{C}(E_A^\infty)$  are called *cohomologous*, if there exists a function  $\psi \in \mathcal{C}(E_A^\infty)$  such that  $f_1 - f_2 = \psi - \psi \circ \sigma$ . A function  $f \in \mathcal{C}(E_A^\infty)$  is said to be *lattice*, if  $f$  is cohomologous to a function whose range is contained in a discrete subgroup of  $\mathbb{R}$ . Otherwise, we say that  $f$  is *non-lattice*. (ii) If the geometric potential function  $\xi$  is non-lattice, then we call the cGDS  $\Phi$  (and inaccurately also  $F$ ) *non-lattice*. On the other hand, if  $\xi$  is lattice, then we call  $\Phi$  (and inaccurately also  $F$ ) *lattice*.

We let  $H(\mu_{-\delta\xi})$  denote the measure theoretical entropy of the shift-map  $\sigma$  with respect to the unique  $\sigma$ -invariant Gibbs measure  $\mu_{-\delta\xi}$  for the potential function  $-\delta\xi$  (see (5.4) for a definition). The unique probability measure  $\nu$  supported on  $F$ , which for all distinct  $e, e' \in E$  satisfies

$$(3.1) \quad \nu(\phi_e(X_{t(e)}) \cap \phi_{e'}(X_{t(e')})) = 0 \quad \text{and} \quad \nu(\phi_e B) = \int_B |\phi'_e|^\delta d\nu$$

for all Borel sets  $B \subseteq X_{t(e)}$  is called the  $\delta$ -conformal measure associated with  $\Phi$ . The statement on the uniqueness and existence is provided in [MU03, Theorem 4.2.9] and goes back to the work of [Pat76, Sul79, DU91].

For a vertex  $v \in V$  we denote the set of edges whose initial and respectively terminal vertex is  $v$  by

$$I_v := \{e \in E \mid i(e) = v\} \quad \text{and} \quad T_v := \{e \in E \mid t(e) = v\}.$$

Moreover, for  $n \in \mathbb{N}$  we set

$$\begin{aligned} I_v^n &:= \{\omega \in E_A^n \mid i(\omega_1) = v\}, & T_v^n &:= \{\omega \in E_A^n \mid t(\omega_n) = v\}, \\ I_v^* &:= \bigcup_{n \in \mathbb{N}} I_v^n, & T_v^* &:= \bigcup_{n \in \mathbb{N}} T_v^n \quad \text{and} \\ I_v^\infty &:= \{\omega \in E_A^\infty \mid i(\omega_1) = v\}. \end{aligned}$$

For a finite word  $\omega \in E_A^*$  the  $\omega$ -cylinder set is defined to be

$$[\omega] := \{u \in E_A^\infty \mid u_i = \omega_i \text{ for } i \in \{1, \dots, n(\omega)\}\}, \text{ in particular } [\emptyset] = E_A^\infty.$$



Fundamentally important objects in our main statements are the primary gaps of  $F$  and their images. These are certain intervals in the complement of the limit set, which are defined in the following way. Set

$$(3.2) \quad L^v := \left\langle \bigcup_{e \in I_v} \pi[e] \right\rangle \setminus \bigcup_{e \in I_v} \langle \pi[e] \rangle,$$

where  $v \in V$  and  $\langle Y \rangle$  denotes the convex hull of a set  $Y \subset \mathbb{R}$ . We let  $n_v$  denote the number of connected components of  $L^v$ . In Proposition 5.2 we show that  $\bigcup_{v \in V} L^v \neq \emptyset$  if  $\lambda^1(F) = 0$ , hence,  $\sum_{v \in V} n_v \geq 1$ . If  $L^v \neq \emptyset$ , we denote the connected components of  $L^v$  by  $L^{v,j}$ , where  $j$  ranges over  $\{1, \dots, n_v\}$  and call the sets  $L^{v,j}$  the *primary gaps* of  $F$ . For every  $\omega \in T_v^*$  we define  $L_\omega^{v,j} := \phi_\omega(L^{v,j})$  and call these sets the *image gaps* of  $F$ .

**3.2. Exposition of the Main Results.** We are now able to present our main results and for this purpose fix the notation which we introduced in Section 3.1. We in particular let  $\Phi := (\phi_e)_{e \in E}$  denote a cGDS with aperiodic irreducible incidence matrix and let  $F$  denote its limit set. We set  $\delta := \dim_M(F)$  and let  $\xi$  denote the geometric potential function associated with  $\Phi$ . Further, we denote by  $H(\mu_{-\delta\xi})$  the measure theoretical entropy of the shift-map  $\sigma$  with respect to the unique shift-invariant Gibbs measure  $\mu_{-\delta\xi}$  for the potential function  $-\delta\xi$  (see Section 5.2).

**Theorem 3.13** (cGDS – Fractal Curvature Measures). *Assume that  $\lambda^1(F) = 0$ . Then the following hold.*

- (i) *The average fractal curvature measures of  $F$  always exist and are both constant multiples of the  $\delta$ -conformal measure  $\nu$  associated with  $F$ , that is*

$$\tilde{C}_0^f(F, \cdot) = \frac{2^{-\delta}c}{H(\mu_{-\delta\xi})} \cdot \nu(\cdot) \quad \text{and} \quad \tilde{C}_1^f(F, \cdot) = \frac{2^{1-\delta}c}{(1-\delta)H(\mu_{-\delta\xi})} \cdot \nu(\cdot),$$

*where the constant  $c$  is given by the well-defined positive and finite limit*

$$(3.3) \quad c := \lim_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta.$$

- (ii) *If  $\xi$  is non-lattice, then both the 0-th and 1-st fractal curvature measures of  $F$  exist and satisfy  $C_k^f(F, \cdot) = \tilde{C}_k^f(F, \cdot)$  for  $k \in \{0, 1\}$ .*
- (iii) *If  $\xi$  is lattice, then there exists a constant  $\bar{c} \in \mathbb{R}$  such that  $\overline{C}_k^f(F, B) \leq \bar{c}$  for every Borel set  $B \subseteq \mathbb{R}$  and  $k \in \{0, 1\}$ . Moreover,  $\underline{C}_k^f(F, \mathbb{R})$  is positive for  $k \in \{0, 1\}$ . If additionally the system  $\Phi$  consists of analytic maps, then neither the 0-th nor the 1-st fractal curvature measure exists.*

Note that items (ii) and (iii) in particular show that the scaling exponents of  $F$  can alternatively be characterised by  $s_0(F) = \sup\{t \in \mathbb{R} \mid \varepsilon^t \lambda^0(\partial F_\varepsilon) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0\}$  and  $s_1(F) = \sup\{t \in \mathbb{R} \mid \varepsilon^t \lambda^1(F_\varepsilon) \rightarrow \infty \text{ as } \varepsilon \rightarrow 0\}$ .

Using the definition of the Minkowski content and Proposition 3.7, we see that the existence of the fractal curvature measures immediately implies the existence of the Minkowski content. Thus, the Minkowski content of  $F$  exists if  $\xi$  is non-lattice. The lattice case is quite interesting with regard to the Minkowski content. A sufficient condition under which the Minkowski content exists in the lattice case is given in item (iii) of the next theorem. Items (i) and (ii) of the following theorem are immediate consequences of Theorem 3.13.

For an  $\alpha$ -Hölder continuous function  $f \in \mathcal{F}_\alpha(E_A^\infty)$  (see Section 5.2) we let  $\nu_f$  denote the unique eigenmeasure with eigenvalue 1 of the dual of the Perron-Frobenius operator for the potential function  $f$  (see Section 5.2).

**Theorem 3.14** (cGDS – Minkowski Content). *Assume that  $\lambda^1(F) = 0$  and let  $c$  denote the constant given in (3.3). Then the following hold.*

(i) *The average Minkowski content of  $F$  exists and is equal to*

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta}c}{(1-\delta)H(\mu_{-\delta\xi})}.$$

(ii) *If  $\xi$  is non-lattice, then the Minkowski content  $\mathcal{M}(F)$  of  $F$  exists and coincides with  $\widetilde{\mathcal{M}}(F)$ .*

(iii) *If  $\xi$  is lattice, then we have that*

$$0 < \underline{\mathcal{M}}(F) \leq \overline{\mathcal{M}}(F) < \infty.$$

*Further, equality in the above equation can be attained. More precisely let  $\zeta, \psi \in \mathcal{C}(E_A^\infty)$  denote two functions satisfying  $\xi - \zeta = \psi - \psi \circ \sigma$ , where the range of  $\zeta$  is contained in a discrete subgroup of  $\mathbb{R}$  and  $a \in \mathbb{R}$  is maximal such that  $\zeta(E_A^\infty) \subseteq a\mathbb{Z}$ . If, for every  $t \in [0, a)$ , we have that*

(3.4)

$$\sum_{n \in \mathbb{Z}} e^{-\delta an} \nu_{-\delta\zeta} \circ \psi^{-1}([na, na+t)) = \frac{e^{\delta t} - 1}{e^{\delta a} - 1} \sum_{n \in \mathbb{Z}} e^{-\delta an} \nu_{-\delta\zeta} \circ \psi^{-1}([na, (n+1)a)),$$

*then  $\underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F)$ .*

**Remark 3.15.** (i) The sums occurring in (3.4) are in fact finite sums.

(ii) Equation (3.4) not only implies the existence of the Minkowski content but also that  $\underline{\mathcal{C}}_0^f(F, \mathbb{R}) = \overline{\mathcal{C}}_0^f(F, \mathbb{R})$  (see the proof of Theorem 3.14(iii)).

An example of a lattice limit set of a cGDS, which satisfies (3.4) and thus is Minkowski measurable, is given in Example 4.4. However, in the special case, when the maps  $\phi_e$  of the cGDS are similarities, (3.4) cannot be satisfied. In this case it even turns out, that the limit set  $F$  is Minkowski measurable if and only if the system is non-lattice. This provides an important extension of the result for self-similar sets given in [LP93, Fal95, LvF06] and is reflected in the following theorem (see Definition 2.8 for the definition of a self-similar set).

**Theorem 3.16** (sGDS – Fractal Curvature Measures). *Suppose that  $\Phi$  is an sGDS. Assume that  $\lambda^1(F) = 0$  and let  $h_{-\delta\xi}$  denote the unique strictly positive eigenfunction with eigenvalue one of the Perron-Frobenius operator for the potential function  $-\delta\xi$  (see Section 5.2). Then, additionally to the statements of Theorem 3.13, the following hold.*

(i) *The constant  $c$  from (3.3) simplifies to the finite sum*

$$c = \sum_{v \in V} \sum_{j=1}^{n_v} h_{-\delta\xi}(\omega^v) |L^{v,j}|^\delta,$$

*which is independent of the choice of  $\omega^v \in I_v^\infty$ .*

(ii) *If  $\xi$  is lattice, then the following holds. For  $k \in \{0, 1\}$  and for every Borel set  $B \subseteq \mathbb{R}$  for which  $F \cap B$  is a non-empty finite union of sets of the form*

$\pi[\omega]$ , where  $\omega \in E_A^*$ , and for which  $F_\varepsilon \cap B = (F \cap B)_\varepsilon$  for all sufficiently small  $\varepsilon > 0$  we have that

$$0 < \underline{C}_k^f(F, B) < \overline{C}_k^f(F, B) < \infty.$$

As an immediate consequence of Theorems 3.14 and 3.16 we obtain the following corollary which we state without a proof.

**Corollary 3.17** (sGDS – Minkowski Content). *Under the conditions of Theorem 3.16 the following hold.*

(i) *The average Minkowski content of  $F$  exists and is given by*

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta} \sum_{v \in V} \sum_{j=1}^{n_v} h_{-\delta\xi}(\omega^v) |L^{v,j}|^\delta}{(1-\delta)H(\mu_{-\delta\xi})},$$

*where the above value is independent of the choice of  $\omega^v \in I_v^\infty$ .*

(ii) *If  $\xi$  is non-lattice, then the Minkowski content  $\mathcal{M}(F)$  of  $F$  exists and is equal to  $\widetilde{\mathcal{M}}(F)$ .*

(iii) *If  $\xi$  is lattice, then*

$$0 < \underline{\mathcal{M}}(F) < \overline{\mathcal{M}}(F) < \infty.$$

The statement that the limit set of an sGDS is Minkowski measurable if and only if it is non-lattice cannot be carried over to general cGDS as we have already seen in Theorem 3.14. Below, we will see that this dichotomy already fails to hold for the subclass of piecewise  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of limit sets of sGDS, where  $\alpha \in (0, 1]$  and  $\mathcal{C}^{1+\alpha}$  denotes the class of real-valued functions which are differentiable with  $\alpha$ -Hölder continuous derivative. However, here there is a dichotomy for the fractal curvature measures. That is, the fractal curvature measures of such an image exist if and only if the underlying system is non-lattice. This is stated in the next theorem, where we moreover provide a relationship between the (average) fractal curvature measures of the limit set of the sGDS and of its piecewise  $\mathcal{C}^{1+\alpha}$ -diffeomorphic image. The analogue statements of Theorem 3.18(i) and (ii) have been obtained in [FK12] for conformal  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of self-similar sets in higher dimensional ambient spaces.

**Theorem 3.18** ( $\mathcal{C}^{1+\alpha}$  Images – Fractal Curvature Measures). *Let  $R$  denote an sGDS with aperiodic irreducible incidence matrix, with associated directed multi-graph  $(V, E, i, t)$  and associated compact non-empty intervals  $(Y_v)_{v \in V}$ . Let  $K \subset \mathbb{R}$  denote the limit set of  $R$  and assume that  $\lambda^1(K) = 0$ . For each  $v \in V$  let  $g_v: W_v \rightarrow \mathbb{R}$  denote a  $\mathcal{C}^{1+\alpha}(W_v)$ -diffeomorphism which is defined on a connected open neighbourhood  $W_v \subset \mathbb{R}$  of  $Y_v$  such that  $|g'_v|$  is bounded away from zero on  $W_v$  and such that the interiors of  $X_v := g_v(Y_v)$  are pairwise disjoint and  $\alpha \in (0, 1]$ . Set  $F := \bigcup_{v \in V} g_v(K \cap Y_v)$ . Then we have the following.*

(i) *The average fractal curvature measures of both  $K$  and  $F$  exist. Moreover,  $\widetilde{C}_k^f(F, \cdot)$  is absolutely continuous with respect to the push-forward measure  $\widetilde{C}_k^f(K, \bigcup_{v \in V} g_v^{-1}(\cdot))$  for  $k \in \{0, 1\}$ . Their Radon-Nikodym derivative is, for  $v \in V$  and  $k \in \{0, 1\}$ , given by*

$$\left. \frac{d\widetilde{C}_k^f(F, \cdot)}{d\widetilde{C}_k^f(K, \bigcup_{v' \in V} g_{v'}^{-1}(\cdot))} \right|_{X_v} = |g'_v \circ g_v^{-1}|^\delta \Big|_{X_v},$$

*where  $\delta := \dim_M(K)$  denotes the Minkowski dimension of  $K$ .*

- (ii) If  $R$  is non-lattice, then the fractal curvature measures of both  $K$  and  $F$  exist and coincide with the respective average fractal curvature measures.
- (iii) If  $R$  is lattice, then neither the 0-th nor the 1-st fractal curvature measure of  $K$  and  $F$  exist.

We have already alluded to the observation that there exist  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of sets arising from lattice sGDS, which are Minkowski measurable. In fact, for every limit set  $K$  of a lattice sGDS there exist  $\mathcal{C}^{1+\alpha}$ -diffeomorphisms  $g$  such that  $g(K)$  is Minkowski measurable. The explicit form of such diffeomorphisms is given in item (iii) of the next theorem.

**Theorem 3.19** ( $\mathcal{C}^{1+\alpha}$ -Images – Minkowski Content). *Suppose that we are in the situation of Theorem 3.18. Let  $\nu$  denote the  $\delta$ -conformal measure associated with  $K$ . Then we have the following.*

- (i) *The average Minkowski content of both  $K$  and  $F$  exist and they are related by*

$$\widetilde{\mathcal{M}}(F) = \widetilde{\mathcal{M}}(K) \cdot \sum_{v \in V} \int_{K \cap Y_v} |g'_v|^\delta d\nu.$$

- (ii) *If  $R$  is non-lattice, then the Minkowski contents of both  $K$  and  $F$  exist and coincide with the respective average Minkowski contents.*
- (iii) *Assume that  $K \subseteq [0, 1]$  and that the geometric potential function  $\zeta$  associated with  $R$  is lattice. Let  $a > 0$  be maximal such that the range of  $\zeta$  is contained in  $a\mathbb{Z}$ . Define  $\tilde{g}: \mathbb{R} \rightarrow \mathbb{R}$ ,  $\tilde{g}(x) := \nu((-\infty, x])$  to be the distribution function of  $\nu$ . For  $n \in \mathbb{N}$  define the function  $g_n: [-1, \infty) \rightarrow \mathbb{R}$  by*

$$g_n(x) := \int_{-1}^x (\tilde{g}(r)(e^{\delta a n} - 1) + 1)^{-1/\delta} dr$$

*and set  $F^n := g_n(K)$ . Then for every  $n \in \mathbb{N}$  we have  $\underline{\mathcal{M}}(F^n) = \overline{\mathcal{M}}(F^n)$ .*

Items (i) and (ii) of the preceding theorem are direct consequences of the respective items in Theorem 3.18 together with Theorem 3.13. The proof of item (iii) has been given in [KK12, Corollary 2.18(iii)] for self-conformal sets. For limit sets of cGDS the proof follows through by using Theorem 3.14(iii) and thus, we are not going to repeat it here.

*Remark 3.20.* The sets  $F^n$  constructed in Theorem 3.19 are actually not only Minkowski measurable but also satisfy  $\underline{C}_0^f(F^n) = \overline{C}_0^f(F^n)$  (see [KK12, Remark 2.19]).

The set  $F$  constructed in Theorem 3.19 is a limit set of a cGDS. Sets of such type play an important role in the theory of general lattice cGDS. Namely, if a lattice cGDS consists of analytic maps, then its limit set  $F$  is an image of a limit set of an sGDS under a piecewise  $\mathcal{C}^{1+\alpha}$ -diffeomorphism:

**Theorem 3.21** (Lattice analytic cGDS). *Let  $\Phi$  be a lattice cGDS consisting of analytic maps and let  $F \subset \mathbb{R}$  denote its limit set. Then there exists a limit set  $K \subset \mathbb{R}$  of a lattice sGDS, with associated non-empty compact intervals  $(Y_v)_{v \in V}$  and  $\mathcal{C}^{1+\alpha}(W_v)$  maps  $g_v: W_v \rightarrow \mathbb{R}$  with  $|g'_v|$  bounded away from zero, where  $W_v$  is an open neighbourhood of  $Y_v$  and  $\alpha \in (0, 1]$ , such that  $F = \bigcup_{v \in V} g_v(K \cap Y_v)$ .*

We end this section with concluding remarks addressing conjectures from [Lap93].

- Remark 3.22* (On two conjectures by Lapidus from 1993). (i) Conjecture 3 in [Lap93] states that under the OSC a non-degenerate self-similar set in  $\mathbb{R}^d$  is Min-kowski measurable if and only if it is non-lattice. This conjecture was proven to be correct in space dimension  $d = 1$  in [LP93, Fal95, LvF06] under the assumption that the feasible open set is connected. For higher dimensional spaces the part concerning the lattice situation is still an open problem. With Corollary 3.17 we have seen that the Minkowski content of a limit set of an sGDS in  $\mathbb{R}$  exists if and only if the sGDS is non-lattice. Thus, Corollary 3.17 shows that [Lap93, Conjecture 3] is also valid for the more general class of limit sets of sGDS in  $\mathbb{R}$  and in this way provides an important extension to the result from [LP93, Fal95, LvF06]. Moreover, Corollary 3.17 also allows to consider self-similar systems where the OSC is satisfied with disconnected feasible open sets (see Section 4.2).
- (ii) In the same paper, [Lap93], a similar conjecture is posed for so-called ‘approximately’ self-similar sets, namely [Lap93, Conjecture 4]. A precise definition of what is meant by an ‘approximately’ self-similar set is not given. However, limit sets of Fuchsian groups of Schottky type are mentioned as examples. Since conformal maps locally behave like similarities, we also view self-conformal sets and limit sets of cGDS as being ‘approximately’ self-similar. Each of the following three paragraphs deals with one of these classes of examples.

Self-conformal sets have already been treated in [KK12]. The results thereof combined with [LP93, Corollary 2.3] provide a negative answer to [Lap93, Conjecture 4] for such sets (see also [KK12, Example 2.20]). Note that [KK12, Theorem 2.12] combined with [LP93, Corollary 2.3] in particular shows that there exist fractal strings with lattice self-conformal boundary for which the asymptotic second term of the eigenvalue counting function  $N(\lambda)$  of the Laplacian (in the sense of [LP93]) is monotonic. We thank M. Lapidus for pointing out this connection to us.

By definition (see Definition 2.8), self-conformal sets are special types of limit sets of cGDS and thus, the results from [KK12] already imply that Minkowski measurability of the limit set of a cGDS is not equivalent to the cGDS being non-lattice. However, Theorem 3.14 shows the validity of one implication, namely that limit sets of non-lattice cGDS are Minkowski measurable. In this context it is worthwhile to observe that we obtain the reverse implication for the fractal curvature measures of piecewise  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of limit sets of sGDS in Theorem 3.18.

Finally, note that limit sets of Fuchsian groups of Schottky type can be represented as limit sets of cGDS (see Section 4.5). It is well known that such limit sets are always non-lattice (see for example [Lal89, Part II]). Combined with [LP93, Corollary 2.3], Theorem 3.14 thus verifies [Lap93, Conjecture 4] for limit sets of Fuchsian groups of Schottky type. This situation will be investigated in more detail in Section 4.5.

#### 4. EXAMPLES OF LIMIT SETS OF CGDS

We now present classes of systems which can be represented by a cGDS and illustrate our results for such systems. We especially focus on sets which cannot be treated with the previously known results from the literature.

**4.1. cGDS derived from a cIFS.** A cIFS  $\Psi := (\psi_1, \dots, \psi_N)$  has got the property that every function  $\psi_i$  can be concatenated with any other function  $\psi_j$  for  $i, j \in \{1, \dots, N\}$ . Here we define a cGDS in that we additionally put transition rules on  $\Psi$ . This is done by defining an  $N \times N$  matrix  $A' := (A'_{i,j})_{i,j \in \{1, \dots, N\}}$  with entries 0, 1 which determines which functions may follow a given function, that is  $A'_{i,j} = 1$  if and only if  $\psi_i \circ \psi_j$  is allowed. The system  $(\Psi, A')$  then gives rise to a cGDS by setting  $V := \{1, \dots, N\}$ ,  $E := \{1, \dots, M\}$ , where  $M := \sum_{i,j=1}^N A'_{i,j}$  and where for all  $v, v' \in V$  with  $A'_{v,v'} = 1$  there exists an edge  $e \in E$  such that  $i(e) = v$  and  $t(e) = v'$ .

**Example 4.1.** For  $i \in \{1, 2, 3\}$  define  $\psi_i: [0, 1] \rightarrow [0, 1]$  by setting  $\psi_1(x) := x/4$ ,  $\psi_2(x) := x/4 + 3/8$  and  $\psi_3(x) := x/4 + 3/4$  and set

$$A' := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

A corresponding sGDS is given by  $V := \{1, 2, 3\}$ ,  $E := \{1, \dots, 6\}$ ,

$$i(e) := \begin{cases} 1 : e \in \{1, 2\} \\ 2 : e = 3 \\ 3 : e \in \{4, 5, 6\}, \end{cases} \quad t(e) := \begin{cases} 1 : e \in \{1, 4\} \\ 2 : e = 5 \\ 3 : e \in \{2, 3, 6\}, \end{cases} \quad A := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$X_v := \psi_v([0, 1])$  for  $v \in V$  and

$$\begin{aligned} \phi_1: X_1 &\xrightarrow{\psi_1} X_1 & \phi_3: X_3 &\xrightarrow{\psi_2} X_2 & \phi_5: X_2 &\xrightarrow{\psi_3} X_3 \\ \phi_2: X_3 &\xrightarrow{\psi_1} X_1 & \phi_4: X_1 &\xrightarrow{\psi_3} X_3 & \phi_6: X_3 &\xrightarrow{\psi_3} X_3. \end{aligned}$$

Here,  $r = 1/4$ . For determining the Minkowski content of the limit set  $F$  of the sGDS, we apply Corollary 3.17 and thus need to find the primary gaps. Observe that

$$\begin{aligned} \langle \pi[1] \rangle &= [0, 1/16], & \langle \pi[2] \rangle &= [3/16, 1/4], & \langle \pi[3] \rangle &= [9/16, 5/8], \\ \langle \pi[4] \rangle &= [3/4, 13/16], & \langle \pi[5] \rangle &= [57/64, 29/32] \quad \text{and} \quad \langle \pi[6] \rangle &= [15/16, 1]. \end{aligned}$$

Thus,

$$L^1 = \underbrace{\left(\frac{1}{16}, \frac{3}{16}\right)}_{=: L^{1,1}}, \quad L^2 = \emptyset \quad \text{and} \quad L^3 = \underbrace{\left(\frac{13}{16}, \frac{57}{64}\right)}_{=: L^{3,1}} \cup \underbrace{\left(\frac{29}{32}, \frac{15}{16}\right)}_{=: L^{3,2}}.$$

The primary gaps  $L^{1,1}$ ,  $L^{3,1}$  and  $L^{3,2}$  are illustrated in Figure 1. Another quantity in the formula of Corollary 3.17 is the eigenfunction  $h_{-\delta\xi}$  of the Perron-Frobenius

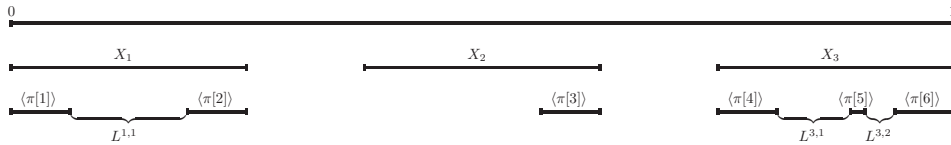


FIGURE 1. Primary gaps of the cGDS from Example 4.1.

operator  $\mathcal{L}_{-\delta\xi}$  (see Section 5.2), where  $\delta$  denotes the Minkowski dimension of  $F$  and  $\xi$  is the geometric potential function associated with  $\Phi$ . In order to determine  $h_{-\delta\xi}$ , we first determine the measure  $\nu_{-\delta\xi}$ . This is done by solving the linear system of equations which arises by combining the following three facts: For  $e \in E$  the defining equation for  $\nu_{-\delta\xi}$  implies that  $\nu_{-\delta\xi}([ee']) = 4^{-\delta} \cdot \nu_{-\delta\xi}([e'])$  for every  $e' \in T_{i(e)}$ ,  $\nu_{-\delta\xi}([e]) = \sum_{e' \in T_{i(e)}} \nu_{-\delta\xi}([ee'])$  and  $\sum_{e \in E} \nu_{-\delta\xi}([e]) = 1$ . The resulting measure  $\nu_{-\delta\xi}$  satisfies

$$\begin{aligned} \nu_{-\delta\xi}([1]) &= \nu_{-\delta\xi}([4]) = (3 \cdot 4^\delta - 4^{-\delta})^{-1}, \\ \nu_{-\delta\xi}([2]) &= \nu_{-\delta\xi}([3]) = \nu_{-\delta\xi}([6]) = (4^\delta - 1) \cdot \nu_{-\delta\xi}([1]) \quad \text{and} \\ \nu_{-\delta\xi}([5]) &= (1 - 4^{-\delta}) \cdot \nu_{-\delta\xi}([1]). \end{aligned}$$

To determine  $h_{-\delta\xi}$ , we use the approximation argument from (5.3). We let  $\mathbf{1}$  denote the constant one-function on  $E_A^\infty$ . Since  $\mathcal{L}_{-\delta\xi}^n \mathbf{1}(u) = \sum_{\omega \in T_v^n} r_\omega^\delta$  for all  $u \in I_v^\infty$  and  $v \in V$ , it follows that  $h_{-\delta\xi}$  is constant on one-cylinders. Now combining the fact that the eigenvalue  $\gamma_{-\delta\xi}$  is equal to one, that  $\mathcal{L}_{-\delta\xi} h_{-\delta\xi} = \gamma_{-\delta\xi} h_{-\delta\xi}$  and that  $\int h_{-\delta\xi} d\nu_{-\delta\xi} = 1$ , we obtain

$$\begin{aligned} h_{-\delta\xi}(\omega^1) &= \frac{3 - 4^{-2\delta}}{-2 \cdot 4^{-\delta} + 6 - 4^\delta} && \text{for } \omega^1 \in I_1^\infty, \\ h_{-\delta\xi}(\omega^2) &= (1 - 4^{-\delta}) \cdot h_{-\delta\xi}(\omega^1) && \text{for } \omega^2 \in I_2^\infty \quad \text{and} \\ h_{-\delta\xi}(\omega^3) &= (4^\delta - 1) \cdot h_{-\delta\xi}(\omega^1) && \text{for } \omega^3 \in I_3^\infty. \end{aligned}$$

From the above evaluations we additionally infer that the Minkowski dimension  $\delta$  is the unique positive root of

$$4^{-\delta} - 4^{-2\delta} + 2 - 4^\delta.$$

Clearly,  $H(\mu_{-\delta\xi}) = \delta \ln 4$ . Thus, altogether we obtain from Corollary 3.17 that

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta} \cdot (3 - 4^{-2\delta})}{(1 - \delta)\delta \ln 4 \cdot (6 - 2 \cdot 4^{-\delta} - 4^\delta)} \left( \left( \frac{1}{8} \right)^\delta + (4^\delta - 1) \left( \left( \frac{5}{64} \right)^\delta + \left( \frac{1}{32} \right)^\delta \right) \right).$$

Since  $\xi = \ln 4$  is lattice, Corollary 3.17 moreover implies that the Minkowski content of  $F$  does not exist.

**4.2. Conformal iterated function systems with disconnected feasible open set.** By definition, a cIFS acting on  $X$  needs to satisfy the OSC with  $\text{int}(X)$  as a feasible open set. If we allow the OSC to be satisfied with a different feasible open set, then the system can still be represented by a cGDS.

**Example 4.2.** For  $i \in \{1, 2, 3\}$  define  $\psi_i: [0, 1] \rightarrow [0, 1]$  by  $\psi_1(x) := x/3$ ,  $\psi_2(x) := x/3 + 2/3$  and  $\psi_3(x) := x/9 + 1/9$  and set  $\Psi := (\psi_1, \psi_2, \psi_3)$ . Then  $\Psi$  is not a cIFS in our sense since the open set condition is not satisfied with  $(0, 1)$  as the feasible open set. (Even though the OSC is satisfied for  $(0, 1/3) \cup (2/3, 1)$ .) However,  $\Psi$  can be represented by an sGDS as follows. Set  $V := \{1, 2\}$ ,  $E := \{1, \dots, 6\}$ ,

$$i(e) := \begin{cases} 1 & : e \in \{1, \dots, 4\} \\ 2 & : e \in \{5, 6\}, \end{cases} \quad t(e) := \begin{cases} 1 & : e \in \{1, 3, 5\} \\ 2 & : e \in \{2, 4, 6\}, \end{cases} \quad A := \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$





FIGURE 2. Primary gaps of the limit set of the cGDS from Example 4.2.

$X_v := \psi_v([0, 1])$  for  $v \in \{1, 2\}$  and

$$\begin{aligned} \phi_1: X_1 &\xrightarrow{\psi_1} X_1, & \phi_3: X_1 &\xrightarrow{\psi_3} X_1, & \phi_5: X_1 &\xrightarrow{\psi_2} X_2, \\ \phi_2: X_2 &\xrightarrow{\psi_1} X_1, & \phi_4: X_2 &\xrightarrow{\psi_3} X_1, & \phi_6: X_2 &\xrightarrow{\psi_2} X_2. \end{aligned}$$

Here,  $r = 1/3$ ,  $L^{1,1} = (4/27, 5/27)$  and  $L^{2,1} = (7/9, 8/9)$ . See Figure 2 for an illustration for this example. That the eigenfunction  $h_{-\delta\xi}$  of the Perron-Frobenius operator  $\mathcal{L}_{-\delta\xi}$  with eigenvalue 1 is equal to the constant one function  $\mathbf{1}$  can be seen as follows. Firstly,  $\mathcal{L}_{-\delta\xi}\mathbf{1} = 2/3^\delta + 1/9^\delta$  and secondly,  $1 = 2/3^\delta + 1/9^\delta$  which can be concluded from the fact that  $0 = P(-\delta\xi)$ , where  $P$  denotes the topological pressure function (see (5.2)). Thus, by Corollary 3.17 we have

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta} \cdot (27^{-\delta} + 9^{-\delta})}{(1-\delta)H(\mu_{-\delta\xi})}.$$

Corollary 3.17 moreover implies that the Minkowski content of  $F$  does not exist, since the range of  $\xi$  is contained in  $\ln 3 \cdot \mathbb{Z}$ .

Alternatively, one can determine the average Minkowski content of this example by using the results of [Gat00]. However, if  $\psi_1, \psi_2$  and  $\psi_3$  were non-linear but conformal, then Theorems 3.13 and 3.14 could be applied, whereas this case is not covered in [Gat00].

**4.3. Markov Interval Maps.** For closed intervals  $X_1, \dots, X_N$  in  $[0, 1]$  with disjoint interior,  $N \geq 2$ , and  $X := \bigcup_{i=1}^N X_i$  we call a map  $g: X \rightarrow [0, 1]$  a *Markov interval map* if

- (i)  $g|_{X_i}$  is expanding and there exists a  $C^{1+\alpha}$ -continuation to a neighbourhood of  $X_i$  and
- (ii) if  $g(X_i) \cap X_j \neq \emptyset$  then  $X_j \subset g(X_i)$  for  $i, j \in \{1, \dots, N\}$ .

For a representation by a cGDS, set  $V := \{1, \dots, N\}$  and for  $v \in V$  define  $G_v := \{v' \in V \mid X_{v'} \subseteq g(X_v)\}$ . For every pair  $(v, v')$ , where  $v \in V$  and  $v' \in G_v$  introduce an edge  $e = e(v, v')$  with  $i(e) = v$  and  $t(e) = v'$ . Set  $E := \{e(v, v') \mid v \in V, v' \in G_v\}$  and define  $\phi_e: X_{t(e)} \rightarrow X_{i(e)}$  by  $\phi_e := (g|_{X_{i(e)}})^{-1}|_{X_{t(e)}}$  for  $e \in E$ . Then the repeller of the Markov interval map coincides with the limit set of the corresponding cGDS.

**Example 4.3.** Set  $X_1 := [0, 1/4]$ ,  $X_2 := [1/4, 1/2]$ ,  $X_3 := [2/3, 1]$  and let the Markov interval map  $g: \bigcup_{i=1}^3 X_i \rightarrow [0, 1]$  be given by  $g|_{X_1}(x) := 5x/2$ ,  $g|_{X_2}(x) := 3x - 1/2$  and  $g|_{X_3}(x) := 3x - 2$ . The graph of the Markov interval map  $g$  is presented in Figure 3. A corresponding sGDS is given by  $V := \{1, 2, 3\}$ ,  $E := \{1, \dots, 7\}$ ,

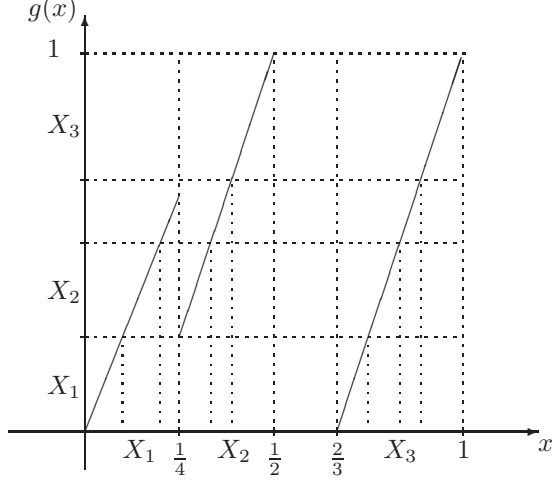


FIGURE 3. Graph of the Markov interval map from Example 4.3.

$$i(e) := \begin{cases} 1 : e \in \{1, 2\} \\ 2 : e \in \{3, 4\} \\ 3 : e \in \{5, 6, 7\}, \end{cases} \quad t(e) := \begin{cases} 1 : e \in \{1, 5\} \\ 2 : e \in \{2, 3, 6\} \\ 3 : e \in \{4, 7\}, \end{cases} \quad A := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} \phi_1: X_1 &\xrightarrow{(g|_{X_1})^{-1}} X_1, \quad \phi_3: X_2 \xrightarrow{(g|_{X_2})^{-1}} X_2, \quad \phi_5: X_1 \xrightarrow{(g|_{X_3})^{-1}} X_3, \\ \phi_2: X_2 &\xrightarrow{(g|_{X_1})^{-1}} X_1, \quad \phi_4: X_3 \xrightarrow{(g|_{X_2})^{-1}} X_2, \quad \phi_6: X_2 \xrightarrow{(g|_{X_3})^{-1}} X_3, \quad \phi_7: X_3 \xrightarrow{(g|_{X_3})^{-1}} X_3. \end{aligned}$$

Here,  $r = 3/4$ . For this example, we limit ourselves to determining and illustrating the primary gaps, since presenting the complete calculations would not provide any further insights. The convex hulls of the projections of the cylinder sets are given by

$$\begin{aligned} \langle \pi[1] \rangle &= [0, 2/25], & \langle \pi[3] \rangle &= [1/4, 1/3], & \langle \pi[5] \rangle &= [2/3, 11/15], \\ \langle \pi[2] \rangle &= [1/10, 1/5], & \langle \pi[4] \rangle &= [7/18, 1/2], & \langle \pi[6] \rangle &= [3/4, 5/6], & \langle \pi[7] \rangle &= [8/9, 1]. \end{aligned}$$

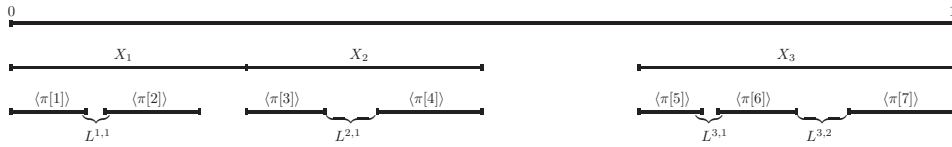


FIGURE 4. Primary gaps for the limit set of Example 4.3.

Thus, the primary gaps are

$$L^{1,1} = (2/25, 1/10), \quad L^{2,1} = (1/3, 7/18), \quad L^{3,1} = (11/15, 3/4) \text{ and } L^{3,2} = (5/6, 8/9).$$

They are illustrated in Figure 4. This cGDS indeed is a non-lattice cGDS and hence the Minkowski content of its limit set exists by Corollary 3.17.

**4.4. Lattice cGDS whose limit set is Minkowski measurable.** An example of a lattice self-conformal set which is Minkowski measurable is given in [KK12, Example 2.20]. In the following, we present an example of a Minkowski measurable limit set of a lattice cGDS which cannot be obtained via a cIFS. This adds to the observations concerning [Lap93, Conjecture 4] that we discussed in Remark 3.22(ii). To be more precise, the following example in conjunction with [LP93, Corollary 2.3] shows the existence of fractal strings, that have a limit set of a lattice cGDS for boundary, for which the asymptotic second term of the eigenvalue counting function  $N(\lambda)$  of the Laplacian (in the sense of [LP93]) is monotonic. This disproves [Lap93, Conjecture 4] for such sets.

**Example 4.4.** Let  $K \subseteq [0, 1]$  denote the limit set of the sGDS given in Example 4.1. Let  $\delta$  denote its Minkowski dimension and let  $\nu$  denote the associated  $\delta$ -conformal measure. Let  $\tilde{g}: \mathbb{R} \rightarrow \mathbb{R}$  denote the distribution function of  $\nu$ , that is  $\tilde{g}(x) := \nu((-\infty, x])$  for  $x \in \mathbb{R}$ . For  $n \in \mathbb{N}$  define the function  $g_n: [-1, \infty) \rightarrow \mathbb{R}$  by

$$g_n(x) := \int_{-1}^x (\tilde{g}(r)(3^{n\delta} - 1) + 1)^{-1/\delta} dr$$

and set  $F^n := g_n(K)$ . Then we have  $\underline{M}(F^n) = \overline{M}(F^n)$ , although  $\underline{M}(K) < \overline{M}(K)$ . This is a consequence of Corollary 3.17 and Theorem 3.19.

**4.5. Limit sets of Fuchsian groups of Schottky type.** Here, we give a very brief introduction to limit sets of Fuchsian groups of Schottky type. For background and proofs of the statements below, we refer the reader to [Nic89, Bea95].

We let  $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$  denote the upper half plane in  $\mathbb{C}$ , where  $\Im(z)$  denotes the imaginary part of  $z \in \mathbb{C}$ . We fix  $n \in \mathbb{N}$  with  $n \geq 2$  and set  $V := \{\pm 1, \dots, \pm n\}$ . We let  $(B_v)_{v \in V}$  denote a family of pairwise disjoint closed Euclidean unit balls in  $\mathbb{C}$  intersecting the real line  $\mathbb{R}$  orthogonally and let  $g_v$  denote the unique hyperbolic conformal orientation preserving automorphism of  $\mathbb{H}$  which maps the side  $s_{-v} := \mathbb{H} \cap \partial B_{-v}$  to the side  $s_v := \mathbb{H} \cap \partial B_v$ . (Note that  $g_v$  is a Möbius transformation which is obtained on concatenating the inversion at the circle  $\partial B_{-v}$  with the reflection at the line  $\Re(z) = d_v$ , where  $d_v = (c_v + c_{-v})/2$  is the midpoint of the line segment joining the centres  $c_{-v}$  and  $c_v$  of the balls  $B_{-v}$  and  $B_v$  and  $\Re(z)$  denotes the real part of  $z \in \mathbb{C}$ .) Then  $\{g_v \mid v \in V\}$  is a symmetric set of generators of the Fuchsian group  $G := \langle \{g_v \mid v \in V\} \rangle$  and  $G$  will be referred to as a Fuchsian group of Schottky type. Associated to  $G$  is a limit set  $L(G) \subset \mathbb{R} \cap \bigcup_{v \in V} B_v$  which is defined to be the set of all accumulation points (with respect to the Euclidean metric on  $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ ) of the  $G$ -orbit  $G(z) := \{g(z) \mid g \in G\}$  for an arbitrary  $z \in \mathbb{H}$ .

Such a limit set can be represented as a limit set of a cGDS in the following way: For defining the directed multigraph we set the set of vertices to be  $V$ , define  $E := \{(v, v') \in V^2 \mid v' \neq -v\}$  to be the set of edges,  $t((v, v')) := v$  and  $i((v, v')) := v'$ . The incidence matrix  $A$  is given by  $A_{e, e'} = 1$  if  $t(e) = i(e')$  and  $A_{e, e'} = 0$  else. It is aperiodic and irreducible, which can be seen as follows. Let  $e, e''' \in E$  denote two arbitrary edges. The condition that  $n \geq 2$  implies that there exist at least two

vertices  $v \in V \setminus \{-t(e), -i(e''')\}$ . Fix  $v$  as such. Since  $v \neq -t(e)$  there exists an edge  $e' \in E$  with  $i(e') = t(e)$  and  $t(e') = v$  and likewise, there exists an edge  $e'' \in E$  with  $i(e'') = v$  and  $t(e'') = i(e''')$ . Thus,  $A_{e,e'''}^{(3)} > 0$ . For  $v \in V$  we set  $X_v := B_v \cap \mathbb{R}$  and note that the maps  $g_v$  can be continuously extended to  $\overline{\mathbb{H}}$ . We denote this extension also by  $g_v$ . For each  $e = (t(e), i(e)) \in E$  we set

$$\phi_e : X_{t(e)} \xrightarrow{g_{i(e)}} X_{i(e)}.$$

Since each  $g_v$  is a Möbius transformation with singularity in  $X_{-v}$ , the map  $\phi_e$  extends to an analytic  $\mathcal{C}^{1+\alpha}$ -diffeomorphism on an open connected neighbourhood  $W_{t(e)}$  of  $X_{t(e)}$ , for some  $\alpha \in (0, 1]$ . Moreover, the maps  $\phi_e$  are strict contractions by construction. That the limit set  $L(G)$  of the Fuchsian group coincides with the limit set of the above constructed cGDS is shown in [MU03, Theorem 5.1.6]. By [Lal89, Part II] the associated geometric potential function is non-lattice. Therefore, we obtain the following corollary from Theorem 3.13:

**Corollary 4.5.** *The fractal curvature measures of a limit set of a Fuchsian group of Schottky type always exist. In particular, a limit set of a Fuchsian group of Schottky type is always Minkowski measurable.*

Note that the above corollary proves [Lap93, Conjecture 4] for limit sets of Fuchsian groups of Schottky type.

**Example 4.6.** In this example we want to show how a typical limit set of a Fuchsian group of Schottky type can be represented as a cGDS. We set  $V := \{\pm 1, \pm 2\}$  and define  $B_{-2}, B_{-1}, B_1$  and  $B_2$  to be the closed unit balls with respective centres  $-5, -2, 2$  and  $5$ . Then the maps  $g_v : \mathbb{H} \rightarrow \mathbb{H}$  are given by

$$g_{-2}(z) = \frac{-5z + 24}{z - 5}, \quad g_{-1}(z) = \frac{-2z + 3}{z - 2}, \quad g_1(z) = \frac{2z + 3}{z + 2} \quad \text{and} \quad g_2(z) = \frac{5z + 24}{z + 5}$$

and  $G := \langle \{g_v \mid v \in V\} \rangle$  is the Fuchsian group of Schottky type. For a representation by a cGDS we set

$$X_{-2} := [-6, -4], \quad X_{-1} := [-3, -1], \quad X_1 := [1, 3] \quad \text{and} \quad X_2 := [4, 6].$$

The set of edges is given by  $E := \{(v, v') \in V^2 \mid v' \neq -v\}$ ,  $t((v, v')) = v$ ,  $i((v, v')) = v'$  and the family of maps  $\phi_e$  for  $e \in E$  is given by

$$\begin{aligned} \phi_{(-2,-2)} : X_{-2} &\xrightarrow{g_{-2}} X_{-2}, & \phi_{(-2,-1)} : X_{-2} &\xrightarrow{g_{-1}} X_{-1}, & \phi_{(-2,1)} : X_{-2} &\xrightarrow{g_1} X_1, \\ \phi_{(-1,-2)} : X_{-1} &\xrightarrow{g_{-2}} X_{-2}, & \phi_{(-1,-1)} : X_{-1} &\xrightarrow{g_{-1}} X_{-1}, & \phi_{(-1,2)} : X_{-1} &\xrightarrow{g_2} X_2, \\ \phi_{(1,-2)} : X_1 &\xrightarrow{g_{-2}} X_{-2}, & \phi_{(1,1)} : X_1 &\xrightarrow{g_1} X_1, & \phi_{(1,2)} : X_1 &\xrightarrow{g_2} X_2, \\ \phi_{(2,-1)} : X_2 &\xrightarrow{g_{-1}} X_{-1}, & \phi_{(2,1)} : X_2 &\xrightarrow{g_1} X_1, & \phi_{(2,2)} : X_2 &\xrightarrow{g_2} X_2. \end{aligned}$$

The incidence matrix  $A$  is a  $12 \times 12$  matrix which contains exactly three ones in every row and every column.

## 5. PRELIMINARIES

We now provide some background information and auxiliary results for proving our main theorems.

### 5.1. Properties of cGDS.

**Proposition 5.1.** *Let  $\Phi$  be a cGDS with limit set  $F$ . Then  $F$  is either a non-empty compact interval or has one-dimensional Lebesgue measure 0.*

*Proof.* For  $n \in \mathbb{N}$  define  $X^{(n)} := \bigcup_{\omega \in E_A^n} \phi_\omega(X_{t(\omega_n)})$  and set  $X := \bigcup_{v \in V} X_v$ . If  $\lambda^1(\text{int}(X) \setminus X^{(1)}) > 0$ , then  $\lambda^1(F) = 0$  by [MU03, Proposition 4.5.9]. Here,  $\text{int}(X)$  denotes the interior of  $X$ . If on the other hand  $\lambda^1(\text{int}(X) \setminus X^{(1)}) = 0$ , then  $\lambda^1(X \setminus X^{(1)}) = 0$ , since the cardinality of  $\partial X$  is finite. It follows that  $X \setminus X^{(1)} = \emptyset$ , since both  $X$  and  $X^{(1)}$  are finite unions of compact intervals. Clearly then  $X^{(n)} = X$  for all  $n \in \mathbb{N}$  and  $F = X$ .  $\square$

**Proposition 5.2.** *Let  $F$  denote the limit set of a cGDS with aperiodic and irreducible incidence matrix (see Definition 2.2). If  $F$  satisfies  $\lambda^1(F) = 0$ , then  $\bigcup_{v \in V} L^v \neq \emptyset$ , where  $L^v$  is defined in (3.2).*

*Proof.* Assume that  $\bigcup_{v \in V} L^v = \emptyset$ . Then  $\bigcup_{v \in V} \langle \bigcup_{e \in I_v} \pi[e] \rangle = \bigcup_{v \in V} \bigcup_{e \in I_v} \langle \pi[e] \rangle$ . This implies

$$\begin{aligned} \Phi \left( \bigcup_{v \in V} \left\langle \bigcup_{e \in I_v} \pi[e] \right\rangle \right) &= \bigcup_{v \in V} \bigcup_{e' \in T_v} \phi_{e'} \left\langle \bigcup_{e \in I_v} \pi[e] \right\rangle = \bigcup_{v \in V} \bigcup_{e' \in T_v} \left\langle \underbrace{\bigcup_{e \in I_v} \pi[e']}_{=\pi[e']} \right\rangle \\ &= \bigcup_{v \in V} \bigcup_{e \in I_v} \langle \pi[e] \rangle = \bigcup_{v \in V} \left\langle \bigcup_{e \in I_v} \pi[e] \right\rangle, \end{aligned}$$

where the second to last equality is due to the fact that the incidence matrix is aperiodic and irreducible. Thus, the set  $\bigcup_{v \in V} \langle \bigcup_{e \in I_v} \pi[e] \rangle$  is invariant under  $\Phi$  and hence  $F = \bigcup_{v \in V} \langle \bigcup_{e \in I_v} \pi[e] \rangle$ . Since we assume that  $\lambda^1(F) = 0$  and since the sets  $\langle \bigcup_{e \in I_v} \pi[e] \rangle$  are compact non-empty intervals, it follows that  $\langle \bigcup_{e \in I_v} \pi[e] \rangle$  is a singleton for every  $v \in V$ . Therefore, the cardinality of  $F$  is finite which contradicts the fact that the Minkowski dimension of  $F$  is positive (see Theorem 5.6).  $\square$

One of the key properties of a cGDS is the bounded distortion property. The following bounded distortion lemma has been obtained in [KK12, Lemma 3.2] in the setting of cIFS. The proof follows along the same lines also for cGDS giving the following lemma.

**Lemma 5.3** (Bounded Distortion). *There exists a sequence  $(\rho_n)_{n \in \mathbb{N}}$  with  $\rho_n > 0$  for all  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \rho_n = 1$  such that for all  $\omega, u \in E_A^*$  with  $u\omega \in E_A^*$  and  $x, y \in \phi_\omega(X_{t(\omega_n(\omega))})$  we have that*

$$\rho_{n(\omega)}^{-1} \leq \frac{|\phi'_u(x)|}{|\phi'_u(y)|} \leq \rho_{n(\omega)}.$$

**5.2. Perron-Frobenius Theory and the Geometric Potential Function.** In order to provide the necessary background to define the constants in our main statement and also to set up the tools needed in the proofs we now recall some facts from the Perron-Frobenius theory.

In the sequel we are going to make use of results from [MU03] which were obtained for finitely primitive conformal graph directed Markov systems (cGDMS), see Remark 2.6. A cGDMS is called finitely primitive, if there exists an  $n \in \mathbb{N}$  such that for all  $e, e' \in E$  there exists an  $\omega \in E_A^n$  for which  $e\omega e' \in E_A^*$ .

*Remark 5.4.* A cGDS with aperiodic irreducible incidence matrix is a finitely primitive cGDMS.

In this subsection we always assume that the incidence matrix  $A$  is aperiodic and irreducible.

Recall the definition of  $E_A^\infty$  from (2.1). We equip  $E_A^\infty$  as defined in (2.1) with the sub-topology of the product topology of the discrete topologies of  $E$  and let  $\mathcal{C}(E_A^\infty)$  denote the set of real-valued continuous functions on  $E_A^\infty$ . For  $f \in \mathcal{C}(E_A^\infty)$ ,  $\alpha \in (0, 1)$  and  $n \in \mathbb{N} \cup \{0\}$  define

$$\text{var}_n(f) := \sup\{|f(\omega) - f(u)| \mid \omega, u \in E_A^\infty \text{ and } \omega_i = u_i \text{ for all } i \in \{1, \dots, n\}\},$$

$$|f|_\alpha := \sup_{n \geq 0} \frac{\text{var}_n(f)}{\alpha^n} \quad \text{and}$$

$$\mathcal{F}_\alpha(E_A^\infty) := \{f \in \mathcal{C}(E_A^\infty) \mid |f|_\alpha < \infty\}.$$

Elements of  $\mathcal{F}_\alpha(E_A^\infty)$  are called  $\alpha$ -Hölder continuous functions on  $E_A^\infty$ .

*Remark 5.5.* The geometric potential function  $\xi$  associated with a cGDS  $\Phi := \{\phi_e\}_{e \in E}$  satisfies  $\xi \in \mathcal{F}_{\tilde{\alpha}}(E_A^\infty)$  for some  $\tilde{\alpha} \in (0, 1)$ . To see this, we let  $r \in (0, 1)$  denote a common Lipschitz constant of  $\phi_e$  for  $e \in E$ . Because of the  $\alpha$ -Hölder continuity of  $\phi'_e$ , we obtain that there exists a constant  $c \in \mathbb{R}$  such that for every  $n \in \mathbb{N}$  we have  $\text{var}_n(\xi) \leq cr^{\alpha(n-1)}$  and  $\text{var}_0(\xi) < \infty$ . Thus,  $\xi \in \mathcal{F}_{\tilde{\alpha}}(E_A^\infty)$ , where  $\tilde{\alpha} := r^\alpha \in (0, 1)$ .

For  $f \in \mathcal{C}(E_A^\infty)$  define the *Perron-Frobenius operator*  $\mathcal{L}_f: \mathcal{C}(E_A^\infty) \rightarrow \mathcal{C}(E_A^\infty)$  by

$$(5.1) \quad \mathcal{L}_f \psi(\omega) := \sum_{u: \sigma u = \omega} e^{f(u)} \psi(u)$$

for  $\omega \in E_A^\infty$  and let  $\mathcal{L}_f^*$  be the dual of  $\mathcal{L}_f$  acting on the set of Borel probability measures on  $E_A^\infty$ . By [Wal01, Theorem 2.16 and Corollary 2.17] and [Bow08, Theorem 1.7], for each real-valued Hölder continuous  $f \in \mathcal{F}_\alpha(E_A^\infty)$ , some  $\alpha \in (0, 1)$ , there exists a unique Borel probability measure  $\nu_f$  on  $E_A^\infty$  such that  $\mathcal{L}_f^* \nu_f = \gamma_f \nu_f$  for some  $\gamma_f > 0$ . Moreover,  $\gamma_f$  is uniquely determined by this equation and satisfies  $\gamma_f = \exp(P(f))$ . Here  $P: \mathcal{C}(E_A^\infty) \rightarrow \mathbb{R}$  denotes the *topological pressure function*, which for  $\psi \in \mathcal{C}(E_A^\infty)$  is defined by

$$(5.2) \quad P(\psi) := \lim_{n \rightarrow \infty} \frac{1}{n} \ln \sum_{\omega \in E_A^n} \exp \sup_{u \in [\omega]} S_n \psi(u),$$

(see [Bow08, Lemma 1.20]), where we recall that  $[\omega] := \{u \in E_A^\infty \mid u_i = \omega_i \text{ for } 1 \leq i \leq n(\omega)\}$  denotes the  $\omega$ -cylinder set and where the  $n$ -th *ergodic sum* of a map  $f: E_A^\infty \rightarrow \mathbb{R}$  and  $n \in \mathbb{N}$  is defined to be

$$S_n f := \sum_{k=0}^{n-1} f \circ \sigma^k \quad \text{and} \quad S_0 f := 0.$$

Further, there exists a unique strictly positive eigenfunction  $h_f \in \mathcal{C}(E_A^\infty)$  of  $\mathcal{L}_f$  satisfying  $\mathcal{L}_f h_f = \gamma_f h_f$ . We take  $h_f$  to be normalised so that  $\int h_f d\nu_f = 1$ . By  $\mu_f$  we denote the  $\sigma$ -invariant probability measure defined by  $\frac{d\mu_f}{d\nu_f} = h_f$ . This is the unique  $\sigma$ -invariant Gibbs measure for the potential function  $f$ . Additionally, under some normalisation assumptions we have convergence of the iterates of the

Perron-Frobenius operator to the projection onto its eigenfunction  $h_f$ . To be more precise we have

$$(5.3) \quad \lim_{m \rightarrow \infty} \|\gamma_f^{-m} \mathcal{L}_f^m \psi - \int \psi d\nu_f \cdot h_f\| = 0 \quad \text{for all } \psi \in \mathcal{C}(E_A^\infty),$$

where  $\|\cdot\|$  denotes the supremum norm on  $\mathcal{C}(E_A^\infty)$ . The results on the Perron-Frobenius operator quoted above originate mainly from the work of Ruelle, see for instance [Rue68].

For the geometric potential function  $\xi \in \mathcal{C}(E_A^\infty)$  it can be shown that the *measure theoretical entropy*  $H(\mu_{-\delta\xi})$  of the shift-map  $\sigma$  with respect to  $\mu_{-\delta\xi}$  is given by

$$(5.4) \quad H(\mu_{-\delta\xi}) = \delta \int_{E_A^\infty} \xi d\mu_{-\delta\xi},$$

where  $\delta$  denotes the Minkowski dimension of  $F$ . This observation follows for example from the variational principle (see [Bow08, Theorem 1.22]) and the following result, which follows by combining [MU03, Theorems 4.2.9, 4.2.11 and 4.2.13].

**Theorem 5.6.** *The Minkowski as well as the Hausdorff dimension of  $F$  is equal to the unique real number  $t > 0$  for which  $P(-t\xi) = 0$ , where  $P$  denotes the topological pressure function.*

**5.3. Renewal Theory and Geometric Measure Theory.** In the proof of Theorem 3.13 we are going to make use of a renewal theory argument for counting measures in symbolic dynamics. For this we call a function  $g_1: (0, \infty) \rightarrow \mathbb{R}$  *asymptotic* to a function  $g_2: (0, \infty) \rightarrow \mathbb{R}$  as  $\varepsilon \rightarrow 0$ , in symbols  $g_1(\varepsilon) \sim g_2(\varepsilon)$  as  $\varepsilon \rightarrow 0$ , if  $\lim_{\varepsilon \rightarrow 0} g_1(\varepsilon)/g_2(\varepsilon) = 1$ . Similarly, we say that  $g_1$  is *asymptotic* to  $g_2$  as  $t \rightarrow \infty$ , in symbols  $g_1(t) \sim g_2(t)$  as  $t \rightarrow \infty$ , if  $\lim_{t \rightarrow \infty} g_1(t)/g_2(t) = 1$ .

The following proposition is a well-known fact which is for example stated in [Lal89, Proposition 2.1].

**Proposition 5.7.** *Let  $f \in \mathcal{F}_\alpha(E_A^\infty)$  for some  $\alpha \in (0, 1)$  be such that for some  $n \geq 1$  the function  $S_n f$  is strictly positive on  $E_A^\infty$ . Then there exists a unique  $s > 0$  such that*

$$(5.5) \quad \gamma_{-sf} = 1.$$

The following two propositions play a crucial role in the proof of Theorem 3.13. The first of the two propositions is [Lal89, Theorem 1]. The second one is a refinement and generalisation of [Lal89, Theorem 3] and proved in [KK12, Theorem 3.9 and Remark 3.10].

**Proposition 5.8** (Lalley). *Assume that  $f$  lies in  $\mathcal{F}_\alpha(E_A^\infty)$  for some  $\alpha \in (0, 1)$ , is non-lattice and such that for some  $n \geq 1$  the function  $S_n f$  is strictly positive. Let  $g \in \mathcal{F}_\alpha(E_A^\infty)$  be non-negative but not identically zero and let  $s > 0$  be implicitly given by (5.5). Then we have that*

$$\sum_{n=0}^{\infty} \sum_{u: \sigma^n u = \omega} g(u) \mathbb{1}_{\{S_n f(u) \leq t\}} \sim \frac{\int g d\nu_{-sf}}{s \int f d\mu_{-sf}} h_{-sf}(\omega) e^{st}$$

as  $t \rightarrow \infty$  uniformly for  $\omega \in E_A^\infty$ .

For  $b \in \mathbb{R}$ , we denote by  $\lceil b \rceil$  the smallest integer which is greater than or equal to  $b$ , by  $\lfloor b \rfloor$  the greatest integer which is less than or equal to  $b$ , and by  $\{b\}$  the fractional part of  $b$ , that is  $\{b\} := b - \lfloor b \rfloor$ .



**Proposition 5.9** (Lalley, Kesseböhmer/Kombrink). *Assume that  $f$  lies in  $\mathcal{F}_\alpha(E_A^\infty)$  for some  $\alpha \in (0, 1)$  and that for some  $n \geq 1$  the function  $S_n f$  is strictly positive. Further assume that  $f$  is lattice and let  $\zeta, \psi \in \mathcal{C}(E_A^\infty)$  denote two functions which satisfy*

$$f - \zeta = \psi - \psi \circ \sigma,$$

where  $\zeta$  is a function whose range is contained in a discrete subgroup of  $\mathbb{R}$ . Let  $a > 0$  be maximal such that  $\zeta(E_A^\infty) \subseteq a\mathbb{Z}$ . Further, let  $g \in \mathcal{F}_\alpha(E_A^\infty)$  be non-negative but not identically zero and  $s > 0$  be implicitly given by (5.5). Then we have that

$$(5.6) \quad \sum_{n=0}^{\infty} \sum_{u: \sigma^n u = \omega} g(u) \mathbb{1}_{\{S_n f(u) \leq t\}} \sim \frac{ah_{-s\zeta}(\omega) \int g(u) e^{-sa \lceil \frac{\psi(u) - \psi(\omega)}{a} - \frac{t}{a} \rceil} d\nu_{-s\zeta}(u)}{(1 - e^{-sa}) \int \zeta d\mu_{-s\zeta}}$$

as  $t \rightarrow \infty$  uniformly for  $\omega \in E_A^\infty$ .

In view of the existence of the average fractal curvature measures the following proposition, which has been obtained in [KK12, Corollary 3.11], is essential.

**Proposition 5.10.** *Under the assumptions of Proposition 5.9*

$$\lim_{T \rightarrow \infty} T^{-1} \int_0^T e^{-st} \sum_{n=0}^{\infty} \sum_{u: \sigma^n u = \omega} g(u) \mathbb{1}_{\{S_n f(u) \leq t\}} dt$$

exists and is equal to

$$\frac{h_{-sf}(\omega) \int g d\nu_{-sf}}{s \int f d\mu_{-sf}}.$$

In order to prove Theorem 3.14(iii), the following lemma which is closely related to Proposition 5.9 is needed (see [KK12, Lemma 3.12]).

**Lemma 5.11.** *Assume the conditions of Proposition 5.9 and fix a non-empty Borel set  $B \subseteq \mathbb{R}$ . For  $\omega \in E_A^\infty$  define the function  $\eta_B: (0, \infty) \rightarrow \mathbb{R}$  by*

$$\eta_B(t) := e^{-st} \int_{E_A^\infty} \mathbb{1}_{\psi^{-1}B}(u) e^{-sa \lceil \frac{\psi(u) - \psi(\omega)}{a} - \frac{t}{a} \rceil} d\nu_{-s\zeta}(u).$$

Then the following are equivalent.

- (i)  $\lim_{t \rightarrow \infty} \eta_B(t)$  exists
- (ii)  $\eta_B$  is constant and
- (iii) for every  $t \in [0, a)$  we have

$$\sum_{n \in \mathbb{Z}} e^{-san} \nu_{-s\zeta \circ \psi^{-1}}(B \cap [na, na+t)) = \frac{e^{st} - 1}{e^{sa} - 1} \sum_{n \in \mathbb{Z}} e^{-san} \nu_{-s\zeta \circ \psi^{-1}}(B \cap [na, (n+1)a)).$$

Another important tool in the proofs of our results is a relationship between the 0-th and the 1-st (average) fractal curvature measures. In order to show that the existence of the 0-th fractal curvature measure implies the existence of the 1-st fractal curvature measure, we use [RW10, Corollary 3.2] which is a higher-dimensional and more general version of the following theorem.

**Proposition 5.12** (Rataj, Winter). *Let  $Y \subset \mathbb{R}$  be a non-empty compact set for which the Minkowski dimension  $\delta := \dim_M(Y)$  exists and which is such that  $\lambda^1(Y) = 0$ . Then*

$$\liminf_{\varepsilon \rightarrow 0} \frac{\varepsilon^\delta \lambda^0(\partial Y_\varepsilon)}{1 - \delta} \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon^{\delta-1} \lambda^1(Y_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^{\delta-1} \lambda^1(Y_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} \frac{\varepsilon^\delta \lambda^0(\partial Y_\varepsilon)}{1 - \delta}.$$

For the results on the average fractal curvature measures we use [RW10, Lemma 4.6(ii)] which is a higher-dimensional version of the next proposition.

**Proposition 5.13** (Rataj, Winter). *Let  $Y \subset \mathbb{R}$  be non-empty and compact and such that its Minkowski dimension  $\delta$  exists and is strictly less than 1. If  $\overline{\mathcal{M}}(Y) < \infty$ , then*

$$\limsup_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2} \lambda^1(Y_\varepsilon) d\varepsilon = (1-\delta)^{-1} \limsup_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(Y_\varepsilon) d\varepsilon \text{ and}$$

$$\liminf_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2} \lambda^1(Y_\varepsilon) d\varepsilon = (1-\delta)^{-1} \liminf_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(Y_\varepsilon) d\varepsilon.$$

## 6. PROOFS OF THEOREMS 3.13 AND 3.14

Theorems 3.13 and 3.14 are generalisations of [KK12, Theorems 2.11 and 2.12] respectively. Their proofs are adaptations of the respective proofs in [KK12]. For convenience of the reader, we now recall the important steps from [KK12] and point out the necessary modifications.

Without loss of generality we assume that  $\{0, 1\} \subset F \subseteq [0, 1]$  as otherwise the result follows by rescaling. We first give the proof for the 0-th fractal curvature measure. Fix an  $\varepsilon > 0$  and consider the expression  $\lambda^0(\partial F_\varepsilon \cap (-\infty, b])/2$  for some  $b \in \mathbb{R}$ . As in [KK12] we express  $\lambda^0(\partial F_\varepsilon \cap (-\infty, b])/2$  in terms of the image gaps but obtain a different representation because of the non-allowed transitions. Note that  $\lambda^0(\partial F_\varepsilon \cap (-\infty, b])$  gives the number of endpoints of the connected components of  $F_\varepsilon$  in  $(-\infty, b]$ , since  $\lambda^0$  is the counting measure.

(6.1)

$$\lambda^0(\partial F_\varepsilon \cap (-\infty, b])/2 = \underbrace{\sum_{v \in V} \sum_{j=1}^{n_v} \#\{\omega \in T_v^* \mid L_\omega^{v,j} \subseteq (-\infty, b], |L_\omega^{v,j}| > 2\varepsilon\}}_{=: \Xi(\varepsilon)} + c_1/2,$$

where  $c_1 \in \{1, 2, 3\}$  depends on the value of  $b$ . For finding appropriate bounds on  $\Xi(\varepsilon)$ , we choose an  $m \in \mathbb{N} \cup \{0\}$  such that all image gaps  $\{L_\omega^{v,j} \mid v \in V, j \in \{1, \dots, n_v\}, \omega \in T_v^m\}$  of level  $m$  are greater than  $2\varepsilon$ . For  $v \in V$ ,  $j \in \{1, \dots, n_v\}$  and  $\omega \in T_v^m$  define

$$\Xi_\omega^{v,j}(\varepsilon) := \#\{u \in T_{i(\omega_1)}^* \mid L_{u\omega}^{v,j} \subseteq (-\infty, b], |L_{u\omega}^{v,j}| > 2\varepsilon\}.$$

We have the following connection.

$$(6.2) \quad \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \Xi_\omega^{v,j}(\varepsilon) \leq \Xi(\varepsilon) \leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \Xi_\omega^{v,j}(\varepsilon) + \sum_{v \in V} n_v \cdot \sum_{j=0}^{m-1} (\#E)^j.$$

For the following, we fix  $b \in \mathbb{R} \setminus F$ . Then  $F \cap (-\infty, b]$  can be expressed as a finite union of sets of the form  $\pi[\kappa]$ , where  $\kappa \in E_A^*$ . To be more precise, let  $l \in \mathbb{N}$  be minimal such that there exist  $\kappa^{(1)}, \dots, \kappa^{(l)} \in E_A^*$  satisfying

- (i)  $F \cap (-\infty, b] = \bigcup_{j=1}^l \pi[\kappa^{(j)}]$  and
- (ii)  $\pi[\kappa^{(i)}] \cap \pi[\kappa^{(j)}]$  contains at most one point for all  $i \neq j$ , where  $i, j \in \{1, \dots, l\}$ .

Then for  $\kappa := \bigcup_{j=1}^l [\kappa^{(j)}]$  the function  $\mathbb{1}_\kappa$  is Hölder continuous. Fix  $\omega \in E_A^m$ . Making use of the existence of the bounded distortion constant  $\rho_{n(\omega)}$  of  $\Phi$  on  $\phi_\omega(X_{t(\omega_{n(\omega)})})$

(see Lemma 5.3), we can give estimates for  $\Xi_\omega^{v,j}(\varepsilon)$ , namely for an arbitrary  $\omega^v \in I_v^\infty$  we have

$$\begin{aligned}
 \Xi_\omega^{v,j}(\varepsilon) &\leq \sum_{n=0}^{\infty} \sum_{u \in T_{i(\omega_1)}^n} \mathbb{1}_\kappa(u\omega\omega^v) \mathbb{1}_{\{|\phi'_u(\pi\omega\omega^v)| \cdot \rho_{n(\omega)} \cdot |L_\omega^{v,j}| > 2\varepsilon\}} + \bar{c}_2(\omega^v, \kappa) \\
 (6.3) \quad &\leq \underbrace{\sum_{n=0}^{\infty} \sum_{u \in T_{i(\omega_1)}^n} \mathbb{1}_\kappa(u\omega\omega^v) \mathbb{1}_{\{|\phi'_u(\pi\omega\omega^v)| \cdot \rho_{n(\omega)} \cdot |L_\omega^{v,j}| \geq 2\varepsilon\}}}_{=: \bar{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa)} + \bar{c}_2(\omega^v, \kappa),
 \end{aligned}$$

where the constant  $\bar{c}_2(\omega^v, \kappa)$  is needed for the following reason.  $L_{u\omega}^{v,j} \subseteq (-\infty, b]$  does not necessarily imply  $u\omega\omega^v \in \kappa$  for an arbitrary  $\omega^v \in I_v^\infty$ . However, if  $n(u) \geq \max_{j=1, \dots, l} n(\kappa^{(j)})$ , either  $[u\omega] \subseteq \kappa$  or  $[u\omega] \cap \kappa = \emptyset$ . Hence, there are only finitely many  $u \in T_{i(\omega_1)}^*$  such that  $L_{u\omega}^{v,j} \subseteq (-\infty, b]$  does not imply that we have  $u\omega\omega^v \in \kappa$  for all  $\omega^v \in I_v^\infty$ . Letting  $\bar{c}_2(\omega^v, \kappa) \in \mathbb{R}$  denote this finite number shows the validity of (6.3) for all  $\varepsilon > 0$ . Likewise, there exists a constant  $\underline{c}_2(\omega^v, \kappa) \in \mathbb{R}$  such that for all  $\varepsilon > 0$

$$\Xi_\omega^{v,j}(\varepsilon) \geq \sum_{n=0}^{\infty} \sum_{u \in T_{i(\omega_1)}^n} \mathbb{1}_\kappa(u\omega\omega^v) \mathbb{1}_{\{|\phi'_u(\pi\omega\omega^v)| \cdot \rho_{n(\omega)}^{-1} \cdot |L_\omega^{v,j}| > 2\varepsilon\}} - \underline{c}_2(\omega^v, \kappa).$$

It follows that for all  $\beta > 1$  we have that

$$\begin{aligned}
 (6.4) \quad \Xi_\omega^{v,j}(\varepsilon) &\geq \underbrace{\sum_{n=0}^{\infty} \sum_{u \in T_{i(\omega_1)}^n} \mathbb{1}_\kappa(u\omega\omega^v) \mathbb{1}_{\{|\phi'_u(\pi\omega\omega^v)| \cdot \rho_{n(\omega)}^{-1} \cdot |L_\omega^{v,j}| \geq 2\varepsilon\beta\}}}_{=: \underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa)} - \underline{c}_2(\omega^v, \kappa). \\
 &=: \underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa)
 \end{aligned}$$

For every  $v \in V$  fix an  $\omega^v \in I_v^\infty$ . Combining (6.1) to (6.4) we obtain that for all  $m \in \mathbb{N}$  and all  $\beta > 1$  we have that

$$(6.5) \quad \bar{\mathcal{C}}_0^f(F, (-\infty, b]) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \bar{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa) \quad \text{and}$$

$$(6.6) \quad \underline{\mathcal{C}}_0^f(F, (-\infty, b]) \geq \liminf_{\varepsilon \rightarrow 0} \varepsilon^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa).$$

In order to prove Theorems 3.13 and 3.14 we want to apply Propositions 5.8 and 5.9 to obtain asymptotics for both the expressions  $\bar{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa)$  and  $\underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa)$ . For this, note that

$$\begin{aligned}
 &\sum_{u \in T_{i(\omega_1)}^n} \mathbb{1}_\kappa(u\omega\omega^v) \cdot \mathbb{1}_{\{|\phi'_u(\pi\omega\omega^v)| \cdot \rho_{n(\omega)}^{\pm 1} \cdot |L_\omega^{v,j}| \geq 2\varepsilon\}} \\
 &= \sum_{u: \sigma^n u = \omega\omega^v} \mathbb{1}_\kappa(u) \cdot \mathbb{1}_{\{\sum_{k=1}^n -\ln|\phi'_{u_k}(\pi\sigma^k u)| \leq -\ln \frac{2\varepsilon}{|L_\omega^{v,j}| \rho_{n(\omega)}^{\pm 1}}\}} \\
 (6.7) \quad &= \sum_{u: \sigma^n u = \omega\omega^v} \mathbb{1}_\kappa(u) \cdot \mathbb{1}_{\{S_n \xi(u) \leq -\ln \frac{2\varepsilon}{|L_\omega^{v,j}| \rho_{n(\omega)}^{\pm 1}}\}}.
 \end{aligned}$$

The hypotheses and Remark 5.5 imply that the geometric potential function  $\xi$  is Hölder continuous and strictly positive. The unique  $s > 0$  for which  $\gamma_{-s\xi} = 1$  is

precisely the Minkowski dimension  $\delta$  of  $F$ , which results by combining the fact that  $\gamma_{-s\xi} = \exp(P(-s\xi))$  for each  $s > 0$  and Theorem 5.6.

Before we distinguish between the lattice and non-lattice case, we prove the following lemma, which is an adaptation of [KK12, Lemma 4.1].

**Lemma 6.1.** *For every  $v \in V$  fix an  $\omega^v \in I_v^\infty$ . Then for an arbitrary  $\Upsilon \in \mathbb{R}$  we have that*

$$(i) \quad \Upsilon \leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} h_{-\delta\xi}(\omega\omega^v) (|L_\omega^{v,j}| \rho_m)^\delta \text{ for all } m \in \mathbb{N} \text{ implies}$$

$$\Upsilon \leq \liminf_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta.$$

$$(ii) \quad \Upsilon \geq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} h_{-\delta\xi}(\omega\omega^v) (|L_\omega^{v,j}| \rho_m^{-1})^\delta \text{ for all } m \in \mathbb{N} \text{ implies}$$

$$\Upsilon \geq \limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta.$$

*Proof.* We are first going to approximate the eigenfunction  $h_{-\delta\xi}$  of the Perron-Frobenius operator  $\mathcal{L}_{-\delta\xi}$ . For that we claim that

$$\mathcal{L}_{-\delta\xi}^n \mathbf{1}(\omega) = \sum_{u \in T_{i(\omega_1)}^n} |\phi'_u(\pi\omega)|^\delta$$

for each  $\omega \in E_A^\infty$  and  $n \in \mathbb{N}$ , where  $\mathbf{1}$  is the constant one-function. This can be easily seen by induction. Since  $\mathcal{L}_{-\delta\xi}^n \mathbf{1}$  converges uniformly to the eigenfunction  $h_{-\delta\xi}$  when taking  $n \rightarrow \infty$  (see (5.3)) we have that

$$\forall t > 0 \exists M \in \mathbb{N}: \forall n \geq M, \forall \omega \in E_A^\infty: \left| \sum_{u \in T_{i(\omega_1)}^n} |\phi'_u(\pi\omega)|^\delta - h_{-\delta\xi}(\omega) \right| < t.$$

Furthermore, through Lemma 5.3 we know that

$$\forall t' > 0 \exists M' \in \mathbb{N}: \forall m \geq M': |\rho_m - 1| < t'.$$

Thus, for all  $n \geq M$  and  $m \geq M'$

$$\begin{aligned} \Upsilon &\leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} h_{-\delta\xi}(\omega\omega^v) (|L_\omega^{v,j}| \rho_m)^\delta \\ &\leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \left( \sum_{u \in T_{i(\omega_1)}^n} |\phi'_u(\pi\omega\omega^v)|^\delta + t \right) |L_\omega^{v,j}|^\delta (1+t')^\delta \\ &\leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \sum_{u \in T_{i(\omega_1)}^n} |L_{u\omega}^{v,j}|^\delta (1+t')^{2\delta} + t(1+t')^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta =: A_{m,n}. \end{aligned}$$

Hence, for all  $t, t' > 0$

$$\begin{aligned} \Upsilon &\leq \liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} A_{m,n} \\ &\leq (1+t')^{2\delta} \liminf_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \sum_{u \in T_{i(\omega_1)}^n} |L_{u\omega}^{v,j}|^\delta \\ &\quad + t(1+t')^\delta \limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta. \end{aligned}$$

We have that  $\sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \leq \sum_{v \in V} n_v \cdot \sum_{\omega \in T_v^m} \|\phi'_\omega\|^\delta =: a_m$ , where  $\|\cdot\|$  denotes the supremum norm. The assertion follows by letting  $t$  and  $t'$  tend to zero, since the sequence  $(a_m)_{m \in \mathbb{N}}$  is bounded by [MU03, Lemma 4.2.12] together with Remark 5.4.

The same arguments can be used to show that a lower bound in the second case is given by  $\limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta$ .  $\square$

### 6.1. The Non-lattice Case.

*Proof of Theorem 3.13(ii).* Even though the proof of Theorem 3.13(ii) follows along the same lines as the proof of [KK12, Theorem 2.11(ii)], we repeat the major steps. Let us fix the notation from the beginning of Section 6.

If  $\mathbb{1}_\kappa$  is identically zero, then  $C_0^f(F, (-\infty, b]) = 0 = \nu(F \cap (-\infty, b])$ . Therefore, in the following, we assume that  $\mathbb{1}_\kappa$  is not identically zero. Combining (6.3), (6.4) and (6.7) with the fact that  $\mathbb{1}_\kappa$  is Hölder continuous allows us to apply Proposition 5.8 to  $\overline{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa)$  and  $\underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa)$ , where  $v \in V$ ,  $j \in \{1, \dots, n_v\}$ ,  $\omega \in T_v^*$ ,  $\omega^v \in I_v^\infty$  and  $\beta > 1$ , and hence gives the following asymptotics.

$$(6.8) \quad \overline{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa) \sim \frac{\nu_{-\delta\xi}(\kappa)}{\delta \int \xi d\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega\omega^v) \cdot (2\varepsilon)^{-\delta} (|L_\omega^{v,j}| \rho_{n(\omega)})^\delta \quad \text{and}$$

$$(6.9) \quad \underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa) \sim \frac{\nu_{-\delta\xi}(\kappa)}{\delta \int \xi d\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega\omega^v) \cdot (2\varepsilon\beta)^{-\delta} (|L_\omega^{v,j}| \rho_{n(\omega)}^{-1})^\delta$$

as  $\varepsilon \rightarrow 0$  uniformly for  $\omega^v \in I_v^\infty$ . Recall that  $H(\mu_{-\delta\xi}) = \delta \int \xi d\mu_{-\delta\xi}$ . On combining (6.5) and (6.8), we obtain for all  $m \in \mathbb{N}$  that

$$\overline{C}_0^f(F, (-\infty, b]) \leq \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} h_{-\delta\xi}(\omega\omega^v) (|L_\omega^{v,j}| \rho_m)^\delta \cdot \nu_{-\delta\xi}(\kappa).$$

Now an application of Lemma 6.1 implies

$$(6.10) \quad \overline{C}_0^f(F, (-\infty, b]) \leq \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \liminf_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \cdot \nu_{-\delta\xi}(\kappa).$$

Analogously, we can conclude that for all  $\beta > 1$

$$\underline{C}_0^f(F, (-\infty, b]) \geq \frac{(2\beta)^{-\delta}}{H(\mu_{-\delta\xi})} \limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \cdot \nu_{-\delta\xi}(\kappa)$$

and hence that

$$(6.11) \quad \underline{C}_0^f(F, (-\infty, b]) \geq \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \cdot \nu_{-\delta\xi}(\kappa).$$

Combining the inequalities (6.10) and (6.11) shows that all the limits occurring therein exist and are equal. Moreover, the  $\delta$ -conformal measure introduced in (3.1) and  $\nu_{-\delta\xi}$  satisfy the relation  $\nu_{-\delta\xi}(\kappa) = \nu((-\infty, b])$ . Therefore,

$$C_0^f(F, (-\infty, b]) = \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \lim_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \cdot \nu(F \cap (-\infty, b])$$

holds for every  $b \in \mathbb{R} \setminus F$ . As  $\mathbb{R} \setminus F$  is dense in  $\mathbb{R}$  the assertion concerning the 0-th fractal curvature measure follows. The result on the 1-st fractal curvature measure now follows by applying Proposition 5.12, as for every  $b \in \mathbb{R} \setminus F$  we have that  $F_\varepsilon \cap (-\infty, b] = (F \cap (-\infty, b])_\varepsilon$  for sufficiently small  $\varepsilon > 0$ .  $\square$

*Proof of Theorem 3.14(ii).* This follows immediately from Theorem 3.13(ii).  $\square$

**6.2. The Lattice Case.** This subsection addresses Theorem 3.13(iii) and Theorem 3.14(iii).

*Proof of Theorem 3.13(iii).* The third statement of Theorem 3.13(iii), namely that neither the 0-th nor the 1-st fractal curvature measure exists if the maps  $\phi_e$  are all analytic, is a direct consequence of Theorem 3.21 together with Theorem 3.18(iii). Thus, we now focus on the boundedness and positivity.

Since  $\xi$  is a lattice function, there exist  $\zeta, \psi \in \mathcal{C}(E_A^\infty)$  such that

$$\xi - \zeta = \psi - \psi \circ \sigma$$

and such that  $\zeta$  is a function whose range is contained in a discrete subgroup of  $\mathbb{R}$ . Let  $a > 0$  be the maximal real number for which  $\zeta(E_A^\infty) \subseteq a\mathbb{Z}$ . Recall from the beginning of Section 6 that the hypotheses and Remark 5.5 imply that  $\xi$  is Hölder continuous and strictly positive and that the unique  $s > 0$  for which  $\gamma_{-s\xi} = 1$  is the Minkowski dimension  $\delta$  of  $F$ .

Fix the notation from the beginning of Section 6. Since  $\mathbf{1}_\kappa$  is Hölder continuous and since we can assume that  $\mathbf{1}_\kappa$  is not identically zero, by combining (6.3), (6.4) and (6.7), we see that an application of Proposition 5.9 to  $\overline{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa)$  and  $\underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa)$  gives the following asymptotics.

$$(6.12) \quad \overline{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa) \sim U_\omega(\omega^v) \int_\kappa e^{-\delta a \left[ \frac{\psi(y) - \psi(\omega\omega^v)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|L_\omega^{v,j}| \rho_n(\omega)} \right]} d\nu_{-\delta\zeta}(y) \quad \text{and}$$

$$(6.13) \quad \underline{A}_\omega^{v,j}(\omega^v, \varepsilon\beta, \kappa) \sim U_\omega(\omega^v) \int_\kappa e^{-\delta a \left[ \frac{\psi(y) - \psi(\omega\omega^v)}{a} + \frac{1}{a} \ln \frac{2\varepsilon\beta\rho_n(\omega)}{|L_\omega^{v,j}|} \right]} d\nu_{-\delta\zeta}(y)$$

as  $\varepsilon \rightarrow 0$  uniformly for  $\omega^v \in I_v^\infty$ , where

$$(6.14) \quad U_\omega(\omega^v) := \frac{ah_{-\delta\zeta}(\omega\omega^v)}{(1 - e^{-\delta a}) \int \zeta d\mu_{-\delta\zeta}}.$$

For the boundedness we first remark that  $\overline{C}_0^f(F, \cdot)$  is monotonically increasing as a set function in the second argument. Therefore, in order to find an upper bound

for  $\overline{C}_0^f(F, \cdot)$  it suffices to consider  $\overline{C}_0^f(F, \mathbb{R})$ . For all  $m \in \mathbb{N}$  we have

$$\begin{aligned}
& \overline{C}_0^f(F, \mathbb{R}) \\
& \stackrel{(6.5)}{\leq} \limsup_{\varepsilon \rightarrow 0} \varepsilon^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \overline{A}_\omega^{v,j}(\omega^v, \varepsilon, E_A^\infty) \\
& \stackrel{(6.12)}{=} \limsup_{\varepsilon \rightarrow 0} \varepsilon^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U_\omega(\omega^v) \int_{E_A^\infty} e^{-\delta a \left[ \frac{\psi(u) - \psi(\omega \omega^v)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|L_\omega^{v,j}| \rho_m} \right]} d\nu_{-\delta\zeta}(u) \\
& \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U_\omega(\omega^v) \int_{E_A^\infty} e^{-\delta a \left( \frac{\psi(u) - \psi(\omega \omega^v)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|L_\omega^{v,j}| \rho_m} \right)} d\nu_{-\delta\zeta}(u) \\
& = \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \frac{a e^{\delta \psi(\omega \omega^v)} h_{-\delta\zeta}(\omega \omega^v)}{(1 - e^{-\delta a}) \int \zeta d\mu_{-\delta\zeta}} \left( \frac{|L_\omega^{v,j}| \rho_m}{2} \right)^\delta \int_{E_A^\infty} e^{-\delta \psi(u)} d\nu_{-\delta\zeta}(u).
\end{aligned}$$

Note that  $h_{-\delta\xi} = e^{\delta\psi} h_{-\delta\zeta}$  and  $d\nu_{-\delta\xi} = e^{-\delta\psi} d\nu_{-\delta\zeta}$ . Hence, by Lemma 6.1 we have that

$$\overline{C}_0^f(F, \mathbb{R}) \leq \liminf_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \frac{a 2^{-\delta}}{(1 - e^{-\delta a}) \int \zeta d\mu_{-\delta\zeta}} =: c_0.$$

$c_0 \in (0, \infty)$  because  $\sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \|\phi'_\omega\|^\delta =: a_m$ , where  $\|\cdot\|$  denotes the supremum norm and the sequence  $(a_m)_{m \in \mathbb{N}}$  is bounded by [MU03, Lemma 4.2.12].

That  $\underline{C}_0^f(F, \mathbb{R})$  is positive can be concluded from the following, where  $\beta > 1$  is arbitrary.

$$\begin{aligned}
& \underline{C}_0^f(F, \mathbb{R}) \\
& \stackrel{(6.6)}{\geq} \liminf_{\varepsilon \rightarrow 0} \varepsilon^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \underline{A}_\omega^{v,j}(\omega^v, \varepsilon \beta, E_A^\infty) \\
& \stackrel{(6.13)}{\geq} \liminf_{\varepsilon \rightarrow 0} \varepsilon^\delta \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U_\omega(\omega^v) \int_{E_A^\infty} e^{-\delta a \left( \frac{\psi(u) - \psi(\omega \omega^v)}{a} + \frac{1}{a} \ln \frac{2\varepsilon \beta \rho_m}{|L_\omega^{v,j}|} + 1 \right)} d\nu_{-\delta\zeta}(u) \\
& = \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \frac{a h_{-\delta\zeta}(\omega \omega^v) e^{\delta \psi(\omega \omega^v) - \delta a}}{(1 - e^{-\delta a}) \int \zeta d\mu_{-\delta\zeta}} \left( \frac{|L_\omega^{v,j}|}{2\beta \rho_m} \right)^\delta \int_{E_A^\infty} e^{-\delta \psi(u)} d\nu_{-\delta\zeta}(u).
\end{aligned}$$

By using  $h_{-\delta\xi} = e^{\delta\psi} h_{-\delta\zeta}$  and  $d\nu_{-\delta\xi} = e^{-\delta\psi} d\nu_{-\delta\zeta}$  and Lemma 6.1, we hence obtain that for all  $\beta > 1$  the following holds.

$$\underline{C}_0^f(F, \mathbb{R}) \geq \limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta \frac{a(2\beta)^{-\delta} e^{-\delta a}}{(1 - e^{-\delta a}) \int \zeta d\mu_{-\delta\zeta}} > 0.$$

The results on  $\underline{C}_1^f(F, B)$  and  $\overline{C}_1^f(F, B)$  are now straightforward applications of Proposition 5.12.  $\square$

*Proof of Theorem 3.14(iii).* The statement on the upper and lower Minkowski contents follows from Theorem 3.13(iii). For the second statement, we use that the



hypotheses and Lemma 5.11 together imply that for every  $v \in V$ ,  $j \in \{1, \dots, n_v\}$ ,  $\omega^v \in I_v^\infty$  and  $\omega \in T_v^m$  we have that

$$\begin{aligned}\overline{U} &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^\delta \int_{E_A^\infty} e^{-\delta a \left[ \frac{\psi(y) - \psi(\omega \omega^v)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|L_\omega^{v,j}| \rho_m} \right]} d\nu_{-\delta\zeta}(y) \cdot \left( \frac{2}{|L_\omega^{v,j}| \rho_m} \right)^\delta \quad \text{and} \\ \underline{U} &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^\delta \int_{E_A^\infty} e^{-\delta a \left[ \frac{\psi(y) - \psi(\omega \omega^v)}{a} + \frac{1}{a} \ln \frac{2\varepsilon \rho_m}{|L_\omega^{v,j}|} \right]} d\nu_{-\delta\zeta}(y) \cdot \left( \frac{2\rho_m}{|L_\omega^{v,j}|} \right)^\delta\end{aligned}$$

are independent of  $\omega$ ,  $v$  and  $j$  and are equal, that is  $\overline{U} = \underline{U} =: U$ . Combining (6.5) and (6.12) and (6.6) and (6.13), where  $\kappa = E_A^\infty$ , we conclude that

$$\begin{aligned}\overline{C}_0^f(F, \mathbb{R}) &\leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U_\omega(\omega^v) \left( \frac{|L_\omega^{v,j}| \rho_m}{2} \right)^\delta \cdot U \quad \text{and} \\ \underline{C}_0^f(F, \mathbb{R}) &\geq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U_\omega(\omega^v) \left( \frac{|L_\omega^{v,j}|}{2\rho_m} \right)^\delta \cdot U,\end{aligned}$$

where  $U_\omega(\omega^v)$  is as defined in (6.14). Applying Lemma 6.1 we obtain  $\overline{C}_0^f(F, \mathbb{R}) = \underline{C}_0^f(F, \mathbb{R})$  and an application of Proposition 5.12 then completes the proof.  $\square$

### 6.3. Average Fractal Curvature Measures.

*Proof of Theorem 3.13(i).* If  $\xi$  is non-lattice, this part immediately follows from Theorem 3.13(ii) and the fact that  $g(\varepsilon) \sim c$  as  $\varepsilon \rightarrow 0$  for some constant  $c \in \mathbb{R}$  implies  $\lim_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{-1} g(\varepsilon) d\varepsilon = c$  for every locally integrable function  $g: (0, \infty) \rightarrow \mathbb{R}$ .

Thus for the rest of the proof we assume that  $\xi$  is lattice and fix the notation from the beginning of Section 6. In particular, recall that  $b \in \mathbb{R} \setminus F$ . We begin with showing the result on the 0-th average fractal curvature measure.

Observe that  $\lim_{T \searrow 0} |\ln T|^{-1} \int_T^1 c \varepsilon^{\delta-1} d\varepsilon = \lim_{T \rightarrow \infty} |T|^{-1} \int_0^T c e^{-\delta t} dt = 0$  for every constant  $c \in \mathbb{R}$ . Fix  $m \in \mathbb{N}$  and define  $M := \min\{|L_\omega^{v,j}| \mid v \in V, j \in \{1, \dots, n_v\}, \omega \in T_v^m\}/2$ . From (6.1) to (6.3) we deduce the following for an arbitrary  $\omega^v \in I_v^\infty$ .

$$\begin{aligned}\overline{D} &:= \limsup_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b]) d\varepsilon \\ &\leq \limsup_{T \searrow 0} |\ln T|^{-1} \left( \int_T^M \varepsilon^{\delta-1} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \overline{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa) d\varepsilon \right. \\ &\quad \left. + \frac{1}{2} \int_M^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b]) d\varepsilon \right).\end{aligned}$$

Local integrability of the integrands implies that we have the following equation for all  $m \in \mathbb{N}$  and  $\omega^v \in I_v^\infty$ .

$$\begin{aligned}
\overline{D} &\leq \limsup_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \overline{A}_\omega^{v,j}(\omega^v, \varepsilon, \kappa) d\varepsilon \\
&= \limsup_{T \rightarrow \infty} T^{-1} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \int_0^T e^{-\delta t} \overline{A}_\omega^{v,j}(\omega^v, e^{-t}, \kappa) dt \\
&\stackrel{(6.7)}{=} \limsup_{T \rightarrow \infty} T^{-1} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \int_0^T e^{-\delta t} \sum_{n=0}^{\infty} \sum_{u: \sigma^n u = \omega \omega^v} \mathbb{1}_\kappa(u) \mathbb{1}_{\{S_n \xi(u) \leq t - \ln \frac{2}{|L_\omega^{v,j}| \rho_m}\}} dt \\
&\leq \limsup_{T \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \left( \frac{|L_\omega^{v,j}| \rho_m}{2} \right)^\delta \frac{T - \ln \frac{2}{|L_\omega^{v,j}| \rho_m}}{T} \left( T - \ln \frac{2}{|L_\omega^{v,j}| \rho_m} \right)^{-1} \\
&\quad \times \int_0^{T - \ln \frac{2}{|L_\omega^{v,j}| \rho_m}} e^{-\delta t} \sum_{n=0}^{\infty} \sum_{u: \sigma^n u = \omega \omega^v} \mathbb{1}_\kappa(u) \mathbb{1}_{\{S_n \xi(u) \leq t\}} dt \\
(6.15) \quad &= \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \left( \frac{|L_\omega^{v,j}| \rho_m}{2} \right)^\delta \frac{h_{-\delta \xi}(\omega \omega^v) \nu_{-\delta \xi}(\kappa)}{\delta \int \xi d\mu_{-\delta \xi}}.
\end{aligned}$$

The last equality is an application of Proposition 5.10. Because (6.15) holds for all  $m \in \mathbb{N}$ , an application of Lemma 6.1 gives

$$\begin{aligned}
&\limsup_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b]) d\varepsilon \\
(6.16) \quad &\leq \frac{2^{-\delta} \nu_{-\delta \xi}(\kappa)}{\delta \int \xi d\mu_{-\delta \xi}} \liminf_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta.
\end{aligned}$$

Analogous estimates give

$$\begin{aligned}
&\liminf_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b]) d\varepsilon \\
&\geq \frac{(2\beta)^{-\delta} \nu_{-\delta \xi}(\kappa)}{\delta \int \xi d\mu_{-\delta \xi}} \limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta
\end{aligned}$$

for all  $\beta > 1$  and hence

$$\begin{aligned}
&\liminf_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b]) d\varepsilon \\
(6.17) \quad &\geq \frac{2^{-\delta} \nu_{-\delta \xi}(\kappa)}{\delta \int \xi d\mu_{-\delta \xi}} \limsup_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_\omega^{v,j}|^\delta.
\end{aligned}$$

(6.16) and (6.17) together imply that for every  $b \in \mathbb{R} \setminus F$

$$\lim_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b]) d\varepsilon = \frac{2^{-\delta} c}{H(\mu_{-\delta \xi})} \nu(F \cap (-\infty, b]),$$

where the constant  $c := \lim_{m \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} |L_{\omega}^{v,j}|^{\delta}$  is well defined. Since  $\mathbb{R} \setminus F$  is dense in  $\mathbb{R}$ , the statement on the 0-th average fractal curvature measure in Theorem 3.13(i) follows.

For the statement on the 1-st average fractal curvature measure, we use Theorem 3.13(iii) which states that  $\overline{C}_0^f(F, (-\infty, b]) < \infty$  for every  $b \in \mathbb{R} \setminus F$ . Proposition 5.12 hence implies that  $\overline{M}(F \cap (-\infty, b]) < \infty$  for every  $b \in \mathbb{R} \setminus F$ . Since for every  $b \in \mathbb{R} \setminus F$  we have that  $(F \cap (-\infty, b])_{\varepsilon} = F_{\varepsilon} \cap (-\infty, b]$  for sufficiently small  $\varepsilon > 0$ , we can thus apply Proposition 5.13 to  $F \cap (-\infty, b]$  and obtain the desired statement.  $\square$

*Proof of Theorem 3.14(i).* This part follows immediately from Theorem 3.13(i).  $\square$

## 7. PROOFS CONCERNING THE SPECIAL CASES – THEOREMS 3.16, 3.18 AND 3.21

In this section, we provide the proofs of the results concerning limit sets of sGDS and piecewise  $\mathcal{C}^{1+\alpha}$ -diffeomorphic images of limit sets of sGDS. In both cases, for showing the statements on the non-existence of the fractal curvature measures, we need the following lemma, which is stated and proven in [KK12, Lemma 5.1].

**Lemma 7.1.** *Let  $F$  denote the limit set of a cGDS  $\Phi := (\phi_e)_{e \in E}$  and let  $\delta$  denote its Minkowski dimension. Further, let  $B \subseteq \mathbb{R}$  denote a Borel set for which  $F_{\varepsilon} \cap B = (F \cap B)_{\varepsilon}$  for all sufficiently small  $\varepsilon > 0$ . Assume that there exists a positive, bounded, periodic and Borel-measurable function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which has the following properties.*

- (i)  *$f$  is not equal to an almost everywhere constant function.*
- (ii) *There exist sequences  $(a_m)_{m \in \mathbb{N}}$  and  $(c_m)_{m \in \mathbb{N}}$ , where  $a_m, c_m > 0$  for all  $m \in \mathbb{N}$  and  $a_m \rightarrow 1$  as  $m \rightarrow \infty$  such that the following property is satisfied. For all  $t > 0$  and  $m \in \mathbb{N}$  there exists an  $M \in \mathbb{N}$  such that for all  $T \geq M$*

$$(7.1) \quad \begin{aligned} & (1-t)a_m^{-\delta} f(T - \ln a_m) - c_m e^{-\delta T} \\ & \leq e^{-\delta T} \lambda^0(\partial F_{e^{-T}} \cap B) \leq (1+t)a_m^{\delta} f(T + \ln a_m) + c_m e^{-\delta T}. \end{aligned}$$

*Then for  $k \in \{0, 1\}$  we have that*

$$\underline{C}_k^f(F, B) < \overline{C}_k^f(F, B).$$

**7.1. sGDS – Proof of Theorem 3.16.** Throughout this section let  $\Phi := (\phi_e)_{e \in E}$  denote an sGDS, thus  $\phi_e$  is a similarity for every  $e \in E$ . Let  $r_e \in (0, 1)$  denote a Lipschitz constant of  $\phi_e$  for  $e \in E$ . Further, set  $r_{\omega} := r_{\omega_1} \cdots r_{\omega_n}$  for a finite word  $\omega = \omega_1 \cdots \omega_n \in E_A^n$ .

*Proof of Theorem 3.16.* We start by showing item (i). This part follows straight from Theorem 3.13(i) with the following considerations. For  $v \in V$  and  $\omega^v \in I_v^{\infty}$ , (5.3) implies

$$(7.2) \quad \lim_{m \rightarrow \infty} \sum_{\omega \in T_v^m} r_{\omega}^{\delta} = \lim_{m \rightarrow \infty} \mathcal{L}_{-\delta\xi}^m \mathbf{1}(\omega^v) = h_{-\delta\xi}(\omega^v).$$

Moreover, since  $\phi_e$  are similarities,  $|L_{\omega}^{v,j}| = r_{\omega} |L^{v,j}|$  for all  $v \in V$ ,  $j \in \{1, \dots, n_v\}$  and  $\omega \in T_v^*$ . Thus,  $c$  from (3.3) simplifies to

$$(7.3) \quad c = \sum_{v \in V} \sum_{j=1}^{n_v} h_{-\delta\xi}(\omega^v) |L^{v,j}|^{\delta},$$

showing the assertion.

In order to prove item (ii) we are going to make use of (6.12) and (6.13).

As  $F \cap B$  has got a representation as a finite non-empty union of sets of the form  $\pi[\omega]$  with  $\omega \in E_A^* \setminus \{\emptyset\}$ , there is a set  $\kappa \subseteq E_A^\infty$  which is a finite union of cylinder sets and which satisfies  $\pi\kappa = F \cap B$ . For this  $\kappa$ ,  $\mathbb{1}_\kappa$  is Hölder continuous. Furthermore, the range of the geometric potential function  $\xi$  of a lattice sGDS itself is contained in a discrete subgroup of  $\mathbb{R}$ . Hence, setting  $\zeta = \xi$  and  $\psi$  equal to a constant function, we have that  $\xi = \zeta + \psi - \psi \circ \sigma$  with functions  $\psi, \zeta \in \mathcal{C}(E_A^\infty)$ , where the range of  $\zeta$  is contained in a discrete subgroup of  $\mathbb{R}$ . Moreover,  $\rho_m = 1$  for all  $m \in \mathbb{N}$  and one easily verifies that  $|L_\omega^{v,j}| = r_\omega |L^{v,j}|$  holds for all  $v \in V$ ,  $j \in \{1, \dots, n_v\}$  and  $\omega \in T_v^*$ . For these reasons the methods from the beginning of Section 6 simplify in the following way.

Let  $T \geq 0$  be sufficiently large such that  $F_{e^{-T}} \cap B = (F \cap B)_{e^{-T}}$  and for  $v \in V$  let  $\omega^v \in I_v^\infty$  be arbitrary. Then there exists a constant  $\tilde{c} \geq 0$ , which depends on the number of sets  $\pi[\omega]$  whose union is  $F \cap B$ , such that

$$\begin{aligned}
 \lambda^0(\partial F_{e^{-T}} \cap B)/2 &\stackrel{(6.1)}{=} \sum_{v \in V} \sum_{j=1}^{n_v} \#\{\omega \in T_v^* \mid L_\omega^{v,j} \subseteq B, |L_\omega^{v,j}| > 2e^{-T}\} + \tilde{c} \\
 &= \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{n=0}^{\infty} \sum_{\omega \in T_v^n} \mathbb{1}_\kappa(\omega \omega^v) \mathbb{1}_{\{|\phi'_\omega(\omega^v)| \cdot |L^{v,j}| > 2e^{-T}\}} + \tilde{c} \\
 &= \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{n=0}^{\infty} \sum_{u: \sigma^n u = \omega^v} \mathbb{1}_\kappa(u) \mathbb{1}_{\{S_n \xi(u) < -\ln \frac{2e^{-T}}{|L^{v,j}|}\}} + \tilde{c} \\
 (7.4) \quad &\sim \sum_{v \in V} \sum_{j=1}^{n_v} \frac{ah_{-\delta\xi}(\omega^v) \nu_{-\delta\xi}(\kappa)}{(1 - e^{-\delta a}) \int \xi d\mu_{-\delta\xi}} \cdot e^{-\delta a \left[ a^{-1} \ln \frac{2e^{-T}}{|L^{v,j}|} \right]} + \tilde{c}
 \end{aligned}$$

as  $T \rightarrow \infty$ , where the last asymptotic is obtained by applying Proposition 5.9. We introduce the function  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  given by

$$f(T) := e^{-\delta T} \frac{a\nu(B)}{(1 - e^{-\delta a})H(\mu_{-\delta\xi})} \sum_{v \in V} \sum_{j=1}^{n_v} h_{-\delta\xi}(\omega^v) e^{-\delta a \left[ \frac{1}{a} \ln \frac{2e^{-T}}{|L^{v,j}|} \right]}.$$

By the asymptotic given in (7.4), we know that for all  $t > 0$  there exists an  $M \in \mathbb{N}$  such that for all  $T \geq M$  we have

$$(1 - t)\delta f(T) \leq e^{-\delta T} \lambda^0(\partial F_{e^{-T}} \cap B)/2 \leq (1 + t)\delta f(T) + ce^{-\delta T}.$$

Clearly,  $f$  is a strictly positive, bounded and periodic function with period  $a$ . Moreover,  $f$  is piecewise continuous with a finite number of discontinuities in an interval of length  $a$ . Additionally, on every interval, where  $f$  is continuous,  $f$  is strictly decreasing. Therefore  $f$  is not equal to an almost everywhere constant function. Thus, all conditions of Lemma 7.1 are satisfied which finishes the proof.  $\square$

**7.2.  $\mathcal{C}^{1+\alpha}$  Images of Limit Sets of sGDS; Proofs of Theorems 3.18, 3.19 and 3.21.** In this subsection we consider the case that  $F$  is the image of the limit set  $K$  of an sGDS under a piecewise  $\mathcal{C}^{1+\alpha}$ -diffeomorphism as in Theorem 3.18. Throughout, we fix the notation from Theorem 3.18. By definition, each  $g_v$  is bi-Lipschitz. Therefore, the Minkowski dimensions of  $K$  and  $F$  coincide (see for instance [Fal03, Corollary 2.4 and Section 3.2]) and are both denoted by  $\delta$ .

The similarities  $(R_e)_{e \in E}$  generating  $K$  and the mappings  $(\phi_e)_{e \in E}$  generating  $F$  are connected through the equations

$$\phi_e = g_{i(e)} \circ R_e \circ g_{t(e)}^{-1}$$

for each  $e \in E$ . We denote by  $\tilde{\pi}$  and  $\pi$  the natural code maps from  $E_A^\infty$  to  $K$  and  $F$  respectively. If we further let  $(r_e)_{e \in E}$  denote the respective similarity ratios of  $(R_e)_{e \in E}$ , we have the following list of observations.

(A) Each map  $\phi_e: X_{t(e)} \rightarrow X_{i(e)}$  is differentiable with derivative

$$\phi'_e = \frac{g'_{i(e)} \circ R_e \circ g_{t(e)}^{-1}}{g'_{t(e)} \circ g_{t(e)}^{-1}} \cdot r_e.$$

- (B) The geometric potential function  $\xi$  associated with  $F$  is given by  $\xi(\omega) = -\ln|g'_{t(\omega_1)}(g_{t(\omega_1)}^{-1}(\pi\omega))| + \ln|g'_{t(\omega_2)}(g_{t(\omega_2)}^{-1}(\pi\sigma\omega))| - \ln r_{\omega_1}$ , where  $\omega = \omega_1\omega_2 \cdots \in E_A^\infty$ . The geometric potential function  $\zeta$  associated with  $K$  is given by  $\zeta(\omega) = -\ln r_{\omega_1}$ . Thus  $\zeta$  is non-lattice, if and only if  $\xi$  is non-lattice.
- (C) The unique  $\sigma$ -invariant Gibbs measure for the potential function  $-\delta\xi$  coincides with the unique  $\sigma$ -invariant Gibbs measure for the potential function  $-\delta\zeta$ , that is  $\mu_{-\delta\xi} = \mu_{-\delta\zeta}$  (see for instance [MU03, Theorem 2.2.7]).
- (D) From items (B) and (C) we obtain that

$$H(\mu_{-\delta\xi}) = \int_{E_A^\infty} \xi d\mu_{-\delta\xi} = \int_{E_A^\infty} \zeta d\mu_{-\delta\zeta} = H(\mu_{-\delta\zeta}).$$

Further, let  $\{\tilde{L}^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  denote the primary gaps of  $K$  and  $\{\tilde{L}_\omega^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  its image gaps for each  $\omega \in E_A^*$  and let  $\{L^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  and  $\{L_\omega^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  respectively denote the primary and the image gaps of  $F$ . Then

(E) The  $\delta$ -conformal measure  $\nu$  associated with  $F$  and the push-forward measure of the  $\delta$ -conformal measure  $\tilde{\nu}$  associated with  $K$  are absolutely continuous with Radon-Nikodym derivative

$$\frac{d\nu}{d\tilde{\nu} \circ g_v^{-1}} \Big|_{X_v} = |g'_v \circ g_v^{-1}|^\delta \Big|_{X_v} \cdot \left( \sum_{v' \in V} \int_{Y_{v'}} |g'_{v'}|^\delta d\tilde{\nu} \right)^{-1}.$$

(F)  $L_\omega^{v,j} = g_{i(\omega_1)}(\tilde{L}_\omega^{v,j})$  for  $v \in V$ ,  $j \in \{1, \dots, n_v\}$  and  $\omega \in T_v^*$ . Define a function  $f: E_A^\infty \rightarrow \mathbb{R}$  by  $f(\omega) := |g'_{i(\omega_1)} \circ \tilde{\pi}(\omega)|^\delta$ . Since  $|\tilde{L}_\omega^{v,j}| = r_\omega |\tilde{L}^{v,j}|$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^n} |L_\omega^{v,j}|^\delta &= \lim_{n \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^n} \left( r_\omega |\tilde{L}^{v,j}| \cdot |g'_{i(\omega_1)}(x^\omega)| \right)^\delta \\ &= \lim_{n \rightarrow \infty} \sum_{v \in V} \sum_{j=1}^{n_v} |\tilde{L}^{v,j}|^\delta \mathcal{L}_{-\delta\zeta}^n(f)(\omega^v) \\ &\stackrel{(5.3)}{=} \sum_{v \in V} \sum_{j=1}^{n_v} |\tilde{L}^{v,j}|^\delta h_{-\delta\zeta}(\omega^v) \cdot \left( \sum_{v' \in V} \int_{Y_{v'}} |g'_{v'}|^\delta d\tilde{\nu} \right), \end{aligned}$$

where  $x^\omega \in \tilde{\pi}[\omega]$  for each  $\omega \in E_A^*$  and  $\omega^v \in I_v^\infty$  for  $v \in V$ . Note that the above equation can be rigorously proven by using the Bounded Distortion Lemma (Lemma 5.3).

- (G) From the fact that  $(R_e)_{e \in E}$  are contractions and each  $g'_v$  is Hölder continuous and bounded away from zero, one can deduce that there exists a cGDS consisting of iterates of  $\Phi := (\phi_e)_{e \in E}$  which all are contractions. As this iterate also generates  $F$ , it follows that  $F$  is a limit set of a cGDS.

*Proof of Theorem 3.18.* Using items (A) to (G) an application of Theorem 3.13(i) and (ii) to  $F$  and of Theorem 3.16 to  $K$  proves Theorem 3.18(i) and (ii).

In order to prove item (iii) we first apply Lemma 7.1 and obtain that there exists a Borel set  $B \subseteq \mathbb{R}$  for which  $\underline{C}_k^f(F, B) < \overline{C}_k^f(F, B)$  for  $k \in \{0, 1\}$ . From this we then deduce that the fractal curvature measures do not exist. For applying Lemma 7.1 we first introduce a family  $\Delta$  of non-empty Borel subsets of  $E_A^\infty$ , where  $E_A^\infty$  denotes the code space associated with  $R$ . For every  $\kappa \in \Delta$  we then construct a pair  $(B(\kappa), f_\kappa)$  which consists of a non-empty Borel set  $B(\kappa) \subseteq \mathbb{R}$  satisfying  $F_\varepsilon \cap B(\kappa) = (F \cap B(\kappa))_\varepsilon$  for all sufficiently small  $\varepsilon > 0$  and a positive bounded periodic Borel-measurable function  $f_\kappa: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , such that Lemma 7.1(ii) is satisfied for  $B = B(\kappa)$  and  $f = f_\kappa$ . Then, we show that there always exists a  $\kappa \in \Delta$  for which  $f_\kappa$  is not equal to an almost everywhere constant function, verifying Lemma 7.1(i).

We write  $R =: (R_e)_{e \in E}$  and let  $r_e \in (0, 1)$  denote a Lipschitz constant of  $R_e$  for  $e \in E$ . Note that  $g_v: Y_v \rightarrow X_v$  is bijective for every  $v \in V$ . For  $e \in E$  define

$$\phi_e := g_{i(e)} \circ R_e \circ g_{t(e)}^{-1}$$

and set  $\Phi := (\phi_e)_{e \in E}$ . From the fact that  $R_e$  is a contraction for every  $e \in E$  and each  $g'_v$  is Hölder continuous and bounded away from zero, one can deduce that there exists an iterate  $\tilde{\Phi}$  of  $\Phi$  which solely consists of contractions. Without loss of generality we assume that  $\phi_e$  are contractions themselves. Then  $\Phi$  is a cGDS and  $F$  is the associated limit set. The code space associated with  $\Phi$  is also  $E_A^\infty$ . We let  $\tilde{\pi}$  and  $\pi$  respectively denote the code maps from  $E_A^\infty$  to  $K$  and  $F$ . They satisfy  $\pi(\omega) = g_{i(\omega_1)} \circ \tilde{\pi}$  for  $\omega \in E_A^\infty$ .

With this notation, we now introduce the family  $\Delta$ . Fix an  $n \in \mathbb{N} \cup \{0\}$  and define

$$\Delta_n := \left\{ \bigcup_{i=1}^l [\kappa^{(i)}] \mid \kappa^{(i)} \in E_A^n, l \in \{1, \dots, \#E_A^n\}, \bigcup_{i=1}^l \langle \tilde{\pi}[\kappa^{(i)}] \rangle \text{ is an interval,} \right. \\ \left. \bigcup_{i=1}^l \tilde{\pi}[\kappa^{(i)}] \cap \tilde{\pi}[\omega] = \emptyset \text{ for every } \omega \in E_A^n \setminus \{\kappa^{(1)}, \dots, \kappa^{(l)}\} \right\}.$$

(Note that if the strong separation condition was satisfied, then  $\Delta_n = \{[\omega] \mid \omega \in E_A^n\}$ .) We remark that the condition  $\lambda^1(K) = 0$  implies that  $\kappa \subsetneq E_A^\infty$  for every  $\kappa \in \Delta_n$ , whenever  $n \in \mathbb{N} \setminus \{0\}$ . Further, note that  $\Delta_n \neq \emptyset$  for all  $n \in \mathbb{N}$  because of the OSC and set  $\Delta := \bigcup_{n \in \mathbb{N} \cup \{0\}} \Delta_n$ . Now, fix an  $n \in \mathbb{N} \cup \{0\}$  and a  $\kappa = \bigcup_{i=1}^l [\kappa^{(i)}] \in \Delta_n$  and choose  $\theta > 0$  such that  $\bigcup_{i=1}^l \langle \tilde{\pi}[\kappa^{(i)}] \rangle_{2\theta} \cap \tilde{\pi}[\omega] = \emptyset$  for every  $\omega \in E_A^n \setminus \{\kappa^{(1)}, \dots, \kappa^{(l)}\}$ . Then  $B(\kappa) := \bigcup_{i=1}^l \langle \tilde{\pi}[\kappa^{(i)}] \rangle_\theta$  is a non-empty Borel subset of  $\mathbb{R}$  satisfying  $F_\varepsilon \cap B(\kappa) = (F \cap B(\kappa))_\varepsilon$  for all  $\varepsilon < \theta$ . We let  $\{L^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  denote the primary gaps of  $F$  and  $\{L_\omega^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  the associated image gaps.

For constructing the function  $f_\kappa$  fix an  $m \in \mathbb{N}$  and choose  $M \in \mathbb{N}$  so that

- (i)  $e^{-M} < \theta$  and that
- (ii)  $|L_\omega^{v,j}| > 2e^{-M}$  holds for all  $v \in V$ ,  $j \in \{1, \dots, n_v\}$  and  $\omega \in T_v^m$  for which  $L_\omega^{v,j} \subset B(\kappa)$ .

Then for all  $T \geq M$  we have

$$\begin{aligned}
 \lambda^0(\partial F_{e^{-T}} \cap B(\kappa)) / 2 &= \sum_{v \in V} \sum_{j=1}^{n_v} \#\{\omega \in T_v^* \mid L_{\omega}^{v,j} \subseteq B(\kappa), |L_{\omega}^{v,j}| > 2e^{-T}\} + 1 \\
 (7.5) \quad &\leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \Xi_{\omega}^{v,j}(e^{-T}) + \underbrace{\sum_{v \in V} n_v \sum_{j=1}^{m-1} (\#E)^{j-1}}_{=: c_m} + 1,
 \end{aligned}$$

where

$$\Xi_{\omega}^{v,j}(e^{-T}) := \#\{u \in T_{i(\omega_1)}^* \mid L_{u\omega}^{v,j} \subseteq B(\kappa), |L_{u\omega}^{v,j}| > 2e^{-T}\}.$$

Likewise

$$\lambda^0(\partial F_{e^{-T}} \cap B(\kappa)) / 2 \geq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \Xi_{\omega}^{v,j}(e^{-T}).$$

The purpose of the following is to give appropriate bounds on  $\Xi_{\omega}^{v,j}(e^{-T})$ . We let  $\xi$  and  $\zeta$  denote the geometric potential functions associated with  $\Phi$  and  $R$ . For  $\omega \in E_A^\infty$  we have the following relation.

$$\begin{aligned}
 \xi(\omega) &= -\ln|\phi'_{\omega_1}(\pi\sigma\omega)| \\
 &= -\ln|g'_{i(\omega_1)}(R_{\omega_1}g_{t(\omega_1)}^{-1}\pi\sigma\omega)| - \ln|R'_{\omega_1}(g_{t(\omega_1)}^{-1}\pi\sigma\omega)| + \ln|g'_{t(\omega_1)}(g_{t(\omega_1)}^{-1}\pi\sigma\omega)| \\
 &= -\ln|g'_{i(\omega_1)}(\tilde{\pi}\omega)| + \zeta(\omega) + \ln|g'_{t(\omega_1)}(\tilde{\pi}\sigma\omega)|.
 \end{aligned}$$

Therefore,  $\psi: E_A^\infty \rightarrow \mathbb{R}$  given by  $\psi(\omega) := -\ln|g'_{i(\omega_1)}(\tilde{\pi}\omega)|$  defines a function lying in  $\mathcal{C}(E_A^\infty)$  which satisfies

$$\xi - \zeta = \psi - \psi \circ \sigma.$$

Let  $c$  be the common Hölder constant of  $g_v$  for  $v \in V$  and let  $k > 0$  be such that for each  $v \in V$  we have that  $|g'_v| \geq k$  on  $W_v$ . We then have the following for all  $x, y \in \langle \tilde{\pi}[\omega] \rangle$ , where  $\omega \in I_v^n$  for  $n \in \mathbb{N}$  and  $v \in V$ .

$$(7.6) \quad \left| \frac{g'_v(x)}{g'_v(y)} \right| \leq \left| \frac{g'_v(x) - g'_v(y)}{g'_v(y)} \right| + 1 \leq \frac{c|x - y|^\alpha}{k} + 1 \leq \max_{\omega \in I_v^n} \frac{c|\langle \pi[\omega] \rangle|^\alpha}{k} + 1 =: p_n.$$

Clearly,  $p_n \rightarrow 1$  as  $n \rightarrow \infty$ . We let  $\{\tilde{L}^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  denote the primary gaps of  $K$  and  $\{\tilde{L}_{\omega}^{v,j}\}_{v \in V, j \in \{1, \dots, n_v\}}$  the image gaps. Further, we let  $\omega^v \in I_v^\infty$  be arbitrary and  $\omega \in T_v^m$ . Then

$$\begin{aligned}
 |L_{u\omega}^{v,j}| &= |g_{i(u_1)}\tilde{L}_{u\omega}^{v,j}| \leq |g'_{i(u_1)}(R_{u\omega}\tilde{\pi}\omega^v)| p_m \cdot |R'_u(R_{\omega}\tilde{\pi}\omega^v)| \cdot |\tilde{L}_{\omega}^{v,j}| \\
 &= |(g_{i(u_1)} \circ R_u)'(R_{\omega}\tilde{\pi}\omega^v)| \cdot p_m |\tilde{L}_{\omega}^{v,j}| \\
 &= |\phi'_u(\phi_{\omega}\pi\omega^v)| \cdot |g'_{i(\omega_1)}(R_{\omega}\tilde{\pi}\omega^v)| \cdot p_m \cdot r_{\omega} |\tilde{L}^{v,j}| \\
 &= \exp(-S_{n(u)}\xi(u\omega\omega^v) - \psi(\omega\omega^v) + \ln(p_m \cdot r_{\omega} |\tilde{L}^{v,j}|)).
 \end{aligned}$$

Therefore, for such  $\omega^v \in I_v^\infty$ ,  $T \geq \max\{M, \tilde{M}\}$  and  $\omega \in T_v^m$  we have that

$$\begin{aligned}
 \Xi_{\omega}^{v,j}(e^{-T}) &\leq \#\{u \in T_{i(\omega_1)}^* \mid L_{u\omega}^{v,j} \subseteq B(\kappa), \\
 &\quad S_{n(u)}\xi(u\omega\omega^v) < -\ln(2e^{-T}) + \ln(p_m r_{\omega} |\tilde{L}^{v,j}|) - \psi(\omega\omega^v)\}.
 \end{aligned}$$

By construction we have  $\mathbf{1}_\kappa \in \mathcal{F}_\alpha(E_A^\infty)$ . By the prerequisites,  $\zeta$  is lattice. Since  $\zeta$  is the geometric potential function associated with an sGDS, the range of  $\zeta$  is



contained in a discrete subgroup of  $\mathbb{R}$ . We let  $a > 0$  denote the maximal real number for which  $\zeta(E_A^\infty) \subseteq a\mathbb{Z}$ . An application of Proposition 5.9 hence yields

$$\begin{aligned} & \lambda^0(\partial F_{e^{-T}} \cap B(\kappa))/2 - c_m \\ & \leq \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \sum_{n=0}^{\infty} \sum_{u: \sigma^n u = \omega \omega^v} \mathbb{1}_\kappa(u) \cdot \mathbb{1}_{\{S_n \xi(u) < -\ln(2e^{-T}) + \ln(p_m r_\omega |\tilde{L}^{v,j}|) - \psi(\omega \omega^v)\}} \\ (7.7) \quad & \sim \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} \frac{a h_{-\delta\zeta}(\omega \omega^v) \int_\kappa e^{-\delta a \left[ \frac{\psi(u) - \psi(\omega \omega^v)}{a} + \frac{1}{a} \ln \frac{2e^{-T}}{p_m r_\omega |\tilde{L}^{v,j}|} + \frac{\psi(\omega \omega^v)}{a} \right]} d\nu_{-\delta\zeta}(u)}{(1 - e^{-\delta a}) \int \zeta d\mu_{-\delta\zeta}}. \end{aligned}$$

Define  $U := a(1 - e^{-\delta a})^{-1} (\int \zeta d\mu_{-\delta\zeta})^{-1}$ . Using that  $\ln r_\omega \in a\mathbb{Z}$  for every  $\omega \in E_A^*$ , the right hand side of (7.7) can be rewritten as

$$\sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U r_\omega^\delta h_{-\delta\zeta}(\omega \omega^v) \int_\kappa e^{-\delta a \left[ \frac{\psi(u)}{a} + \frac{1}{a} \ln \frac{2e^{-T}}{p_m |\tilde{L}^{v,j}|} \right]} d\nu_{-\delta\zeta}(u).$$

Defining the function  $f_\kappa: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  by

$$f_\kappa(T) := e^{-\delta T} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U h_{-\delta\zeta}(\omega \omega^v) r_\omega^\delta \int_\kappa e^{-\delta a \left[ \frac{\psi(u)}{a} + \frac{1}{a} \ln \frac{2e^{-T}}{|\tilde{L}^{v,j}|} \right]} d\nu_{-\delta\zeta}(u)$$

we thus have that for all  $t > 0$  there exists a  $\tilde{M} \in \mathbb{R}$  such that

$$e^{-\delta T} \lambda^0(\partial F_{e^{-T}} \cap B(\kappa))/2 \leq (1+t) p_m^\delta f_\kappa(T + \ln p_m) + c_m e^{-\delta T}.$$

for all  $T \geq \tilde{M}$  and likewise,

$$e^{-\delta T} \lambda^0(\partial F_{e^{-T}} \cap B(\kappa))/2 \geq (1-t) p_m^{-\delta} f_\kappa(T - \ln p_m).$$

Clearly,  $f_\kappa$  is periodic with period  $a$ . Thus, Lemma 7.1(ii) is satisfied for  $B = B(\kappa)$  and  $f = f_\kappa$ .

We now prove the validity of Lemma 7.1(i), that is that there exists a  $\kappa \in \Delta$  for which  $f_\kappa$  is not equal to an almost everywhere constant function. Recall that  $\{y\}$  denotes the fractional part of  $y \in \mathbb{R}$ . Set  $\underline{\beta} := \min\{\{a^{-1} \ln |\tilde{L}^{v,j}|\} \mid v \in V, j \in \{1, \dots, n_v\}\}$  and  $\overline{\beta} := \max\{\{a^{-1} \ln |\tilde{L}^{v,j}|\} \mid v \in V, j \in \{1, \dots, n_v\}\}$ . We first assume that  $\underline{\beta} > 0$  and consider the following four cases.

CASE 1:  $\underline{D} := \{\omega \in E_A^\infty \mid \{a^{-1} \psi(\omega)\} < \underline{\beta}\} \neq \emptyset$ .

Since  $\psi \in \mathcal{C}(E_A^\infty)$  and thus  $\underline{D}$  is open, there exists a  $\kappa \in \Delta$  such that  $\kappa \subseteq \underline{D}$ . For  $n \in \mathbb{N}$  and  $r \in (0, 1 - \overline{\beta})$  define  $T_n(r) := a(n+r) + \ln 2$ . Then

$$f_\kappa(T_n(r)) = e^{-\delta ar} \cdot 2^{-\delta} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U h_{-\delta\zeta}(\omega \omega^v) r_\omega^\delta \int_\kappa e^{-\delta a \left[ \frac{\psi(u)}{a} \right] + \delta a} d\nu_{-\delta\zeta}(u).$$

This shows that  $f_\kappa$  is strictly decreasing on  $(an + \ln 2, a(n+1 - \overline{\beta}) + \ln 2)$  for every  $n \in \mathbb{N}$ . Therefore,  $f_\kappa$  is not equal to an almost everywhere constant function.

CASE 2:  $\overline{D} := \{\omega \in E_A^\infty \mid \{a^{-1} \psi(\omega)\} > \overline{\beta}\} \neq \emptyset$ .

Like in CASE 1, there exists a  $\kappa \in \Delta$  such that  $\kappa \subseteq \overline{D}$ . For  $n \in \mathbb{N}$  and  $r \in (0, \underline{\beta})$  set  $T_n(r) := a(n-r) + \ln 2$ . Then

$$f_\kappa(T_n(r)) = e^{\delta ar} \cdot 2^{-\delta} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U h_{-\delta\zeta}(\omega \omega^v) r_\omega^\delta \int_\kappa e^{-\delta a \left[ \frac{\psi(u)}{a} \right]} d\nu_{-\delta\zeta}(u).$$

This shows that  $f_\kappa$  is strictly decreasing on  $(a(n - \underline{\beta}) + \ln 2, an + \ln 2)$  for every  $n \in \mathbb{N}$ . Therefore,  $f_\kappa$  is not equal to an almost everywhere constant function.

For the remaining cases we let  $q^* \in \mathbb{N} \cup \{0\}$  be maximal such that  $\underline{\beta} + q^*(1 - \overline{\beta}) \leq \overline{\beta}$ .

CASE 3: There exists a  $q \in \{0, \dots, q^*\}$  such that

$$D_q := \{\omega \in E_A^\infty \mid \underline{\beta} + q(1 - \overline{\beta}) < \{a^{-1}\psi(\omega)\} < \underline{\beta} + (q+1)(1 - \overline{\beta})\} \neq \emptyset.$$

As in the above cases, there exists a  $\kappa \in \Delta$  such that  $\kappa \subseteq D_q$ . For  $n \in \mathbb{N}$  and  $r \in (0, \underline{\beta})$  set  $T_n^q(r) := a(n - \overline{\beta} + \underline{\beta} + q(1 - \overline{\beta}) - r) + \ln 2$ . Then

$$\begin{aligned} f_\kappa(T_n^q(r)) \\ = e^{\delta ar} 2^{-\delta} e^{\delta a(\overline{\beta} - \underline{\beta} - q(1 - \overline{\beta}))} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U h_{-\delta \zeta}(\omega \omega^v) r_\omega^\delta \int_\kappa e^{-\delta a \lceil \frac{\psi(u)}{a} \rceil} d\nu_{-\delta \zeta}(u). \end{aligned}$$

This shows that  $f_\kappa$  is strictly decreasing on  $(a(n - \overline{\beta} + q(1 - \overline{\beta})) + \ln 2, a(n - \overline{\beta} + \underline{\beta} + q(1 - \overline{\beta})) + \ln 2)$ . Therefore,  $f_\kappa$  is not equal to an almost everywhere constant function.

If neither of the cases 1-3 obtains, then the following case obtains.

CASE 4:  $\{\omega \in E_A^\infty \mid \{a^{-1}\psi(\omega)\} \subseteq \{\underline{\beta} + q(1 - \overline{\beta}) \mid q \in \{0, \dots, q^*\}\}\} = E_A^\infty$ .

Define  $q_i := \min(\{\underline{\beta} + q(1 - \overline{\beta}) - \{a^{-1} \ln |\tilde{L}_{\omega'}^i|\} > 0 \mid q \in \{0, \dots, q^*\}, \omega' \in E_A^{\tilde{M}}\} \cup \{1\})$  and  $p := \min\{q_1, \dots, q_N, 1 - \overline{\beta} + \underline{\beta}\}$ . For  $n \in \mathbb{N}$  and  $r \in (0, p/2)$  define  $T_n(r) := a(n + r) + \ln 2$ . Then

$$\begin{aligned} f_{E_A^\infty}(T_n(r)) \\ = e^{-\delta ar} \cdot 2^{-\delta} \sum_{v \in V} \sum_{j=1}^{n_v} \sum_{\omega \in T_v^m} U h_{-\delta \zeta}(\omega \omega^v) r_\omega^\delta \int_{E_A^\infty} e^{-\delta a \lceil \frac{\psi(u)}{a} - \frac{1}{a} \ln |\tilde{L}_{\omega'}^{v,j}| \rceil} d\nu_{-\delta \zeta}(u). \end{aligned}$$

This shows that  $f_{E_A^\infty}$  is strictly decreasing on  $(an + \ln 2, a(n + p/2) + \ln 2)$ . Therefore,  $f_{E_A^\infty}$  is not equal to an almost everywhere constant function.

If  $\beta = 0$ , then the same methods can be applied after shifting the origin by  $(1 - \overline{\beta})/2$  to the left.

Thus, we can apply Lemma 7.1 in all four cases and obtain that there always exists a Borel set  $B(\kappa)$  such that  $\underline{C}_k^f(F, B(\kappa)) < \overline{C}_k^f(F, B(\kappa))$  for  $k \in \{0, 1\}$ .

In order to deduce that the fractal curvature measures do not exist, construct a function  $\eta: \mathbb{R} \rightarrow [0, 1]$  which is continuous, equal to 1 on  $B(\kappa)$  and equal to 0 on  $\mathbb{R} \setminus B(\kappa)_\theta$ . Then  $\liminf_{\varepsilon \rightarrow 0} \int \eta \varepsilon^\delta d\lambda^0(\partial F_\varepsilon \cap \cdot)/2 = \underline{C}_0^f(F, B(\kappa)) < \overline{C}_0^f(F, B(\kappa)) = \limsup_{\varepsilon \rightarrow 0} \int \eta \varepsilon^\delta d\lambda^0(\partial F_\varepsilon \cap \cdot)/2$ . Thus, the 0-th fractal curvature measure does not exist. Using the same function  $\eta$  it follows analogously, that the 1-st fractal curvature measure does not exist, which completes the proof.  $\square$

*Proof of Theorem 3.21.* We define an operator  $\tilde{\mathcal{L}}: \mathcal{C}(\mathbb{R}) \rightarrow \mathcal{C}(\mathbb{R})$  by setting

$$\tilde{\mathcal{L}}(g)(x) := \sum_{e \in T_v} |\phi'_e(x)|^\delta \cdot g \circ \phi_e(x)$$

for  $x \in X_v$  and  $v \in V$ . Letting  $\xi$  denote the geometric potential function associated with  $\Phi$  and letting  $\pi$  denote the code map from the code space  $E_A^\infty$  to  $F$ , we see that  $\tilde{\mathcal{L}}(g)(\pi\omega) = \mathcal{L}_{-\delta\xi}(g \circ \pi)(\omega)$ , where  $\delta$  denotes the Minkowski dimension of  $F$ .

Since the maps  $\phi_e$  are all analytic, there exist open neighbourhoods  $W_v \supset X_v$  of  $X_v$  in  $\mathbb{C}$  on which the maps  $\phi_e$  are analytic for  $e \in T_v$ . By [MU03, Lemma

4.2.12] the functions  $\tilde{\mathcal{L}}^n \mathbf{1}|_{W_v}$  are uniformly bounded and the bound is independent of  $n \in \mathbb{N}$ . Thus, for  $v \in V$ ,  $\tilde{\mathcal{L}}^n \mathbf{1}: W_v \rightarrow \mathbb{C}$  form a normal family in the sense of Montel. By (5.3) we have that  $\tilde{\mathcal{L}}^n \mathbf{1} \circ \pi$  converges uniformly to  $h_{-\delta\xi}$  on  $E_A^\infty$ . Therefore,  $\tilde{\mathcal{L}}^n \mathbf{1}|_{W_v}$  converges to an analytic extension of  $h_{-\delta\xi}$  on  $W_v$ . We denote this analytic extension by  $h^v$  and set  $\tilde{\psi}_v := \delta^{-1} \ln h^v$ . Since  $\xi$  is lattice, there exist  $\zeta, \psi \in \mathcal{C}(E_A^\infty)$  such that

$$\xi - \zeta = \psi - \psi \circ \sigma$$

and such that the range of  $\zeta$  is contained in a discrete subgroup of  $\mathbb{R}$ . We let  $a > 0$  denote the maximal real number such that  $\zeta(E_A^\infty) \subset a\mathbb{Z}$ .  $\tilde{\psi}_v$  satisfies

$$\tilde{\psi}_v \circ \pi|_{I_v} = \psi|_{I_v} + \delta^{-1} \ln h_{-\delta\xi}|_{I_v},$$

as  $h^v$  satisfies

$$h^v \circ \pi|_{I_v} = h_{-\delta\xi}|_{I_v} = \frac{d\mu_{-\delta\xi}}{d\nu_{-\delta\xi}} \Big|_{I_v} = \frac{d\mu_{-\delta\zeta}}{e^{-\delta\psi} d\nu_{-\delta\zeta}} \Big|_{I_v} = e^{\delta\psi} h_{-\delta\zeta}|_{I_v}.$$

We define  $X_v := [a_v, b_v]$  for  $v \in V$  and introduce the functions  $\tilde{g}_v: [a_v, b_v] \rightarrow \mathbb{R}$  given by

$$\tilde{g}_v(x) := \int_{a_v}^x e^{\tilde{\psi}_v(y)} dy / D_v + 2v,$$

where  $D_v := \int_{a_v}^{b_v} e^{\tilde{\psi}_v(y)} dy$ . Note that  $\tilde{g}_v([a_v, b_v]) = [2v, 2v+1]$ . As  $\tilde{\psi}^v$  is analytic by definition, the fundamental theorem of calculus implies that  $\tilde{g}'_v(x) = e^{\tilde{\psi}_v(x)} / D_v$ , giving

$$\ln \tilde{g}'_v = \tilde{\psi}_v - \ln D_v.$$

Furthermore, the analyticity of  $\tilde{\psi}_v$  implies that  $\tilde{\psi}_v$  is bounded on  $X_v$ . Therefore,  $\tilde{g}'_v$  is bounded away from both 0 and  $\infty$  and hence  $\tilde{g}_v$  is invertible. Set

$$g_v: [2v, 2v+1] \rightarrow [a_v, b_v], \quad g_v := \tilde{g}_v^{-1}$$

and extend  $g_v$  to an analytic function on an open neighbourhood  $\mathcal{U}_v$  of  $[2v, 2v+1]$  such that  $|g'_v| > 0$  on  $\mathcal{U}_v$ . For  $e \in E$  we define

$$R_e := g_{i(e)}^{-1} \circ \phi_e \circ g_{t(e)}$$

and introduce the code map  $\tilde{\pi}$  given by  $\tilde{\pi}|_{I_v} := g_v^{-1} \circ \pi$  for  $v \in V$ . For  $\omega \in E_A^\infty$  we then have

$$\begin{aligned} & -\ln R'_{\omega_1}(\tilde{\pi}\sigma\omega) \\ &= -\ln \tilde{g}'_{i(\omega_1)}(\phi_{\omega_1} g_{t(\omega_1)} \tilde{\pi}\sigma\omega) - \ln \phi'_{\omega_1}(g_{t(\omega_1)} \tilde{\pi}\sigma\omega) + \ln \tilde{g}'_{t(\omega_1)}(g_{t(\omega_1)} \tilde{\pi}\sigma\omega) \\ &= -\tilde{\psi}_{i(\omega_1)}(\pi\omega) + \xi(\omega) + \tilde{\psi}_{t(\omega_1)}(\pi\sigma\omega) + \ln D_{i(\omega_1)} - \ln D_{t(\omega_1)} \\ &= -\psi(\omega) - \delta^{-1} \ln(h_{-\delta\zeta}(\omega)/h_{-\delta\zeta}(\sigma\omega)) + \psi(\sigma\omega) + \xi(\omega) + \ln(D_{i(\omega_1)}/D_{t(\omega_1)}) \\ &= \zeta(\omega) - \delta^{-1} \ln \frac{h_{-\delta\zeta}(\omega)}{h_{-\delta\zeta}(\sigma\omega)} + \ln \frac{D_{i(\omega_1)}}{D_{t(\omega_1)}}. \end{aligned}$$

Since the range of  $\zeta$  is contained in the group  $a\mathbb{Z}$  and  $\xi$  and  $\psi$  are bounded on  $E_A^\infty$ ,  $\zeta$  in fact takes a finite number of values. The continuity of  $\zeta$  implies that there exists an  $M \in \mathbb{N}$  such that  $\zeta$  is constant on each cylinder set  $[\omega]$  for  $\omega \in E_A^M$ . This clearly implies that  $\mathcal{L}_{-\delta\zeta}^n \mathbf{1}$  is constant on  $[\omega]$  for all  $\omega \in E_A^M$  and all  $n \in \mathbb{N}$ , where  $\mathbf{1}$  denotes the constant one function. Thus, (5.3) implies that also  $h_{-\delta\zeta}$  is constant on cylinder sets of length  $M$ . This can be seen by considering  $|h_{-\delta\zeta}(\omega) - h_{-\delta\zeta}(u)|$  for

$u, \omega$  lying in the same cylinder set of length  $M$  and applying the triangle inequality. Therefore,  $\omega \mapsto -\ln|R'_{\omega_1}(\tilde{\pi}\sigma\omega)|$  is constant on cylinder sets of length  $M+1$ . Since for each  $\omega \in E_A^{M+1}$  the set  $\{\tilde{\pi}u \mid u \in [\omega]\}$  has accumulation points and is compact and the map  $R'_e$  is analytic by construction, it follows that  $R'_e$  is constant on its domain of definition. Therefore, the maps  $R_e$  are similarities. From the fact that  $\phi_e$  are contractions and each of the  $g'_v$  is differentiable and bounded away from zero, one can deduce that there exists an iterate  $\tilde{R}$  of  $R := (R_e)_{e \in E}$  which solely consists of contractions and thus is an sGDS.  $\square$

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(Marc Kesseböhmer) UNIVERSITÄT BREMEN, BIBLIOTHEKSTRASSE 1, 28359 BREMEN, GERMANY  
*E-mail address:* `mhk@math.uni-bremen.de`

(Sabrina Kombrink) UNIVERSITÄT BREMEN, BIBLIOTHEKSTRASSE 1, 28359 BREMEN, GERMANY  
*E-mail address:* `kombrink@math.uni-bremen.de`